SEPARATION OF VARIABLES

(*Calculus* 8th ed. pages 421–422 / 9th ed. pages 421–422)

Separation of variables is a strategy of rewriting a differential equation so that each variable occurs on only one side of the equation. After separating the variables, you then integrate both sides of the equation with respect to the appropriate variable. A "constant" can be added to both sides of the equation, but then the constants are combined on one side to make a single constant value; thus you will often see it added to only one side of the equation when the integration step is completed.

For example, $\frac{dy}{dx} = \frac{2x}{y}$. Multiplying both sides by y and by dx gives $y\,dy = 2x\,dx$.

Integrating both sides, $\int y\,dy = \int 2x\,dx$ leads to $\frac{y^2}{2} = x^2 + C_1$, and thus $y^2 - 2x^2 = C$.

(*Note:* You cannot really multiply by dx as it has a symbolic meaning. However, this allows us to show the integration with respect to the correct variable. If we look at it from the perspective of the Chain Rule, where y is a function of x, $dy = y'\,dx$, we begin the problem as $y' = \frac{2x}{y}$, multiply both sides by y, integrate with respect to x, and we get $\int yy'dx = \int 2x\,dx$. Then make the dy substitution and you end up with $\int y\,dy = \int 2x\,dx$.)

DIFFERENTIAL EQUATIONS: GROWTH AND DECAY

(*Calculus* 8th ed. pages 413–417 / 9th ed. pages 415–419)

The separation of variables technique is often used for growth and decay problems, where the rate of change (derivative) of a variable y is proportional to the value of y.

Consider $\frac{dy}{dt} = ky$, where the derivative (rate of change) of y with respect to t (time) is proportional to y (where k is the constant of proportionality).

The subsequent exponential model that is derived from the above equation is $y = Ce^{kt}$, where C is the initial value of y and k is the constant of proportionality.

When $k > 0$, exponential growth occurs; when $k < 0$, exponential decay occurs.

The proof of this is an important model to understand in terms of separation of variables. Things to remember:

- When you integrate $1/y$, it is $\ln|y|$.
- When you exponentiate both sides (that is, raise e to the power of both sides of the equation), you get e^{kt+c_1} on the right side of the equation. Using rules of exponents, that becomes $e^{kt}e^{c_1}$,

and e^{c_1} is just a different way of writing a constant. Thus, the right-hand side becomes Ce^{kt}.

Thus, all solutions of the equation $y' = ky$ have the form $y = Ce^{kt}$.

AP Tips

- The applications of this form of equation can include population growth, declining sales or other business applications, and science applications like Newton's Law of Cooling.
- Be sure to set up the equation correctly. For example, with Newton's Law of Cooling, the statement could be that the rate of change of y is proportional to the difference between y and the ambient temperature (for example, 60) and subsequently you would use separation of variables after writing the equation as $y' = k(y - 60)$.
- Required skills for differential equations:
  - ☐ Verify that a function is or is not a solution to a given differential equation.
  - ☐ Recognize exponential growth and decay and the appropriate equations.
  - ☐ Solve a given separable differential equation.
  - ☐ Solve for a particular solution.

LOGISTIC DIFFERENTIAL EQUATIONS (BC TOPIC ONLY)

(*Calculus* 8th ed. pages 427–428 / 9th ed. pages 429–430)

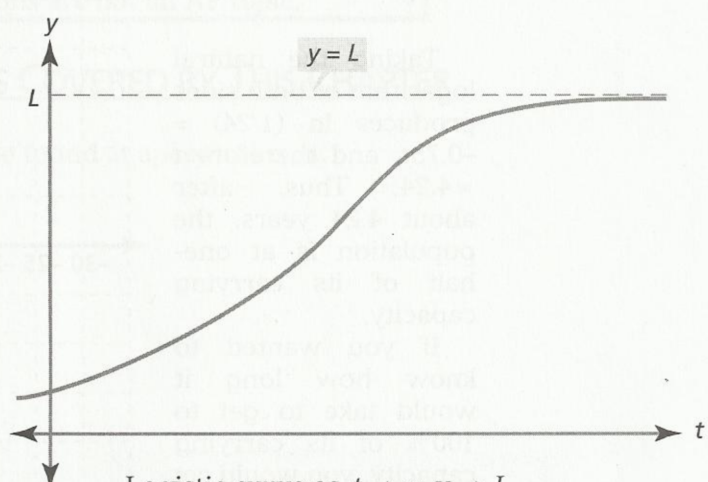
Exponential growth is unlimited. There are instances, however, when exponential growth can be used to model the first portion of a population cycle which levels off to a finite upper limit L. This maximum population L or $y(t)$ that can be sustained or supported as time t increases is called the carrying capacity.

A logistic differential equation $\frac{dy}{dt} = ky(1 - \frac{y}{L})$ is a model that is often used for this type of growth, where k and L are positive constants. If a population satisfies this equation, it approaches the carrying capacity, L, as t increases; it does not grow without bound.

If y is between 0 and L, then $\frac{dy}{dt} > 0$, and the population increases.

If $k > L$, then $\frac{dy}{dt} < 0$, and the population decreases.

After applying the separation of variables' techniques to the logistic differential equation and using partial fractions to integrate, the general solution is of the form



Logistic curve as $t \to \infty, y \to L$.

$y = \frac{L}{1+be^{-kt}}$, $b = \frac{L - y(0)}{y(0)}$ by letting $t = 0$ and solving for b.

Also, note that the maximum rate of growth occurs at $\frac{L}{2}$.

The main process in the solution of a problem is

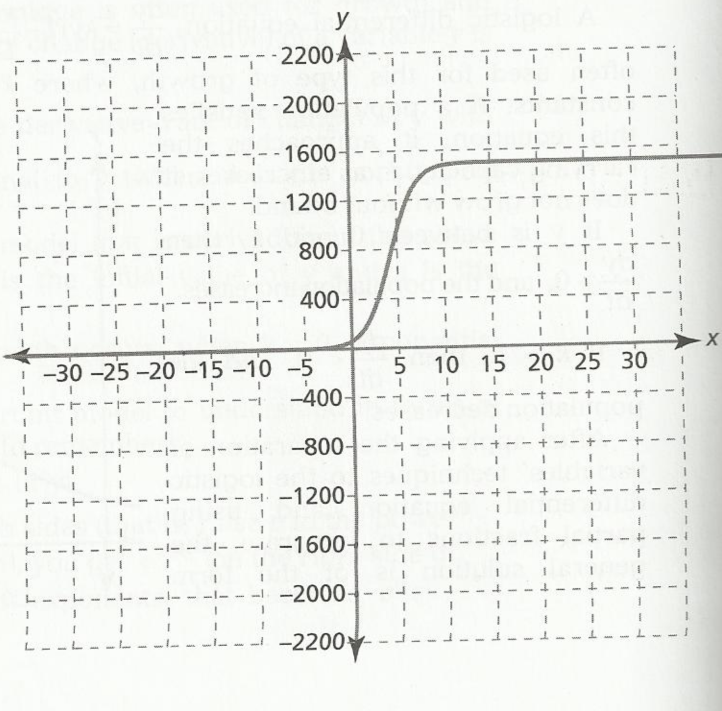
1. Take the model as written as a rate in the form of the general solution, and substitute an appropriate variable for y.
2. Input any known constant(s).
3. Solve for the remaining constant(s) by substituting known information (a value at a given time, for example).
4. Rewrite the solution with the correct variables and constants.
5. Answer a specific question such as the population's value at a specific time.
6. Take the limit as t approaches infinity to determine the carrying capacity.

Let's look at some examples.

a) Try to interpret $P(t) = \frac{1500}{1+24e^{-0.75t}}$.

- L (carrying capacity) = 1500 units (this comes directly from the form $y = \frac{L}{1+be^{-kt}}$).
- $k = 0.75$ (constant part of the exponent of e).
- Initial population is when $t = 0$ (t in years); therefore = $\frac{1500}{1+24e^{0}}$ = 1500/25 = 60 units.
- To determine when the population will reach 50% of its carrying capacity, let $P(t) = \frac{1500}{1+24e^{-0.75t}}$ = 750 and solve for t. Therefore $2 = 1 + 24e^{-0.75t}$, or $1/24 = e^{-0.75t}$.

Taking the natural logarithm of both sides produces $\ln(1/24) = -0.75t$, and therefore $t \approx 4.24$. Thus, after about 4.24 years, the population is at one-half of its carrying capacity.



If you wanted to know how long it would take to get to 100% of its carrying capacity, you would set $P(t) = 1500$. However, this won't work because you would get $0 = e^{-0.75t}$. Therefore, let's take the limit as t approaches infinity to see what happens.

As $t \to \infty$ in $P(t) = \dfrac{1500}{1+24e^{-0.75t}}$, then $\lim\limits_{t\to\infty} = \dfrac{1500}{1+24e^{-0.75t}} = 1500$ (the carrying capacity) because $e^{-0.75t}$ approaches 0.

And finally, solve for the logistic differential equation that has a solution of $P(t) = \dfrac{1500}{1+24e^{-0.75t}}$.

Start with the growth rate equation $\dfrac{dP}{dt} = kP\left(1-\dfrac{P}{L}\right)$, where P is the population at a given time, k is the constant, and L is the carrying capacity.

Then substitute in the known values, and the solution is $\dfrac{dP}{dt} = 0.75P\left(1-\dfrac{P}{1500}\right)$.

b) Now let's start with the logistic differential equation and, given an initial condition, solve for the logistic equation.

$\dfrac{dy}{dt} = y\left(1-\dfrac{y}{40}\right)$, initial condition is (0, 8).

Therefore, at time $t = 0$, the population is $8y(0)$.

We know that $L = 40$ and $k = 1$ from the form of the equation $\dfrac{dP}{dt} = kP\left(1-\dfrac{P}{L}\right)$. (*Note:* y is equivalent to P.) Solving for b in $y = \dfrac{L}{1+be^{-kt}}$, we know that $b = \dfrac{L-y(0)}{y(0)} = \dfrac{40-8}{8} = 4$.

Therefore, $y = \dfrac{40}{1+4e^{-t}}$ is the final solution, by substitution.

AP Tips

- Homogeneous differential equations are *not* an AP topic.
- First-order linear differential equations are *not* an AP topic.

PAST AP FREE-RESPONSE PROBLEMS COVERED BY THIS CHAPTER

Note: These and other questions can be found at apcentral.com.

1998 BC 4
1999 BC 6b
2000 BC 6
2001 BC 5b, c
2002 BC 5
2004 BC 5
2005 AB 6; BC 4
2006 AB 5; BC 5
2007 AB 4b
2008 AB 5; BC 6

MULTIPLE-CHOICE QUESTIONS

A calculator may not be used for the following questions.

An asterisk (*) indicates the question is for BC students.

1. The general solution to the differential equation $\frac{dy}{dx} = y^2 \sin x$ is

(A) $y = \sqrt[3]{3\cos x + C}$

(B) $y = -\cos x + C$

(C) $y = \sqrt[3]{\sin x + C}$

(D) $y = \frac{1}{\cos x} + C$

(E) $y = \sqrt[3]{-2\sec x + C}$

2. If $e^y \frac{dy}{dx} = 2x$ and $y(1) = 2,$ then the particular solution $y(x)$ is

(A) $y = \ln(x^2) + 2.$

(B) $y = \ln(x^2 + e^2 - 1).$

(C) $y = 2e^{x^2-1}.$

(D) $y = x^2 + e^2 - 1.$

(E) $y = \ln(x^2 + e - 4).$

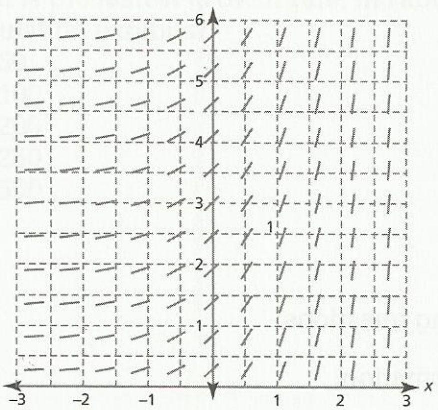
Questions 3–5 refer to the following information:

Consider the differential equation

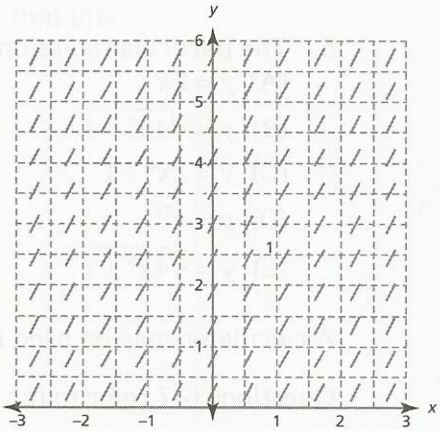
$\frac{dy}{dx} = \frac{4x}{y}$, for $y \geq 1$ only, with initial value $y(0) = 1$.

3. Which of the following is the slope field for the general solution to the given differential equation?

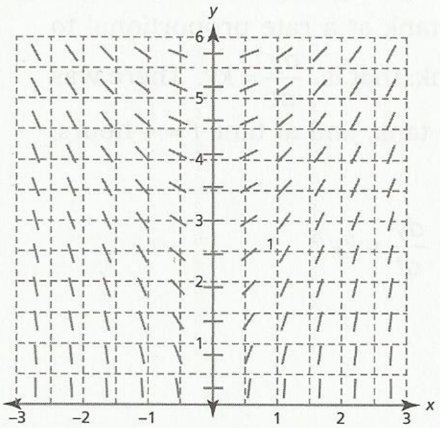
(A)



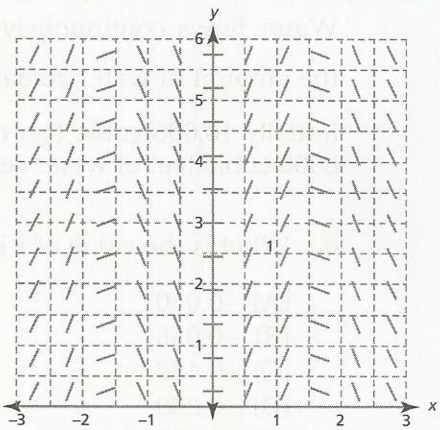
(D)



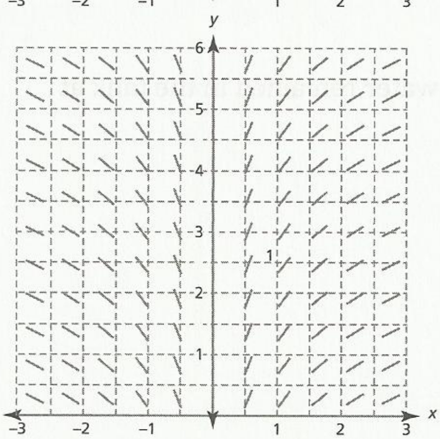
(B)



(E)



(C)



*4. Using Euler's Method with step size $\Delta x = 1/2$, what is the estimate for $y(1)$?
(A) 1
(B) 2
(C) $\sqrt{5}$
(D) 5/2
(E) 3

5. The particular solution $y(x)$ is
(A) $y = 2x$
(B) $y = \sqrt{4x^2 - 4}$
(C) $y = 2x^2 + 1$
(D) $y = e^{2x^2}$
(E) $y = \sqrt{4x^2 + 1}$

A calculator may be used for the following questions.

Questions 6–7 refer to the following information:

Water flows continuously from a large tank at a rate proportional to the amount of water remaining in the tank; that is, $\frac{dy}{dt} = ky$. There was initially 10,000 cubic feet of water in the tank, and at time $t = 4$ hours, 8000 cubic feet of water remained.

6. What is the value of k in the equation $\frac{dy}{dt} = ky$?
(A) –0.050
(B) –0.056
(C) –0.169
(D) –0.200
(E) –0.223

7. To the nearest cubic foot, how much water remained in the tank at time $t = 8$ hours?
(A) 5778
(B) 6000
(C) 6400
(D) 6458
(E) 6619

Questions 8–10 refer to the following information:

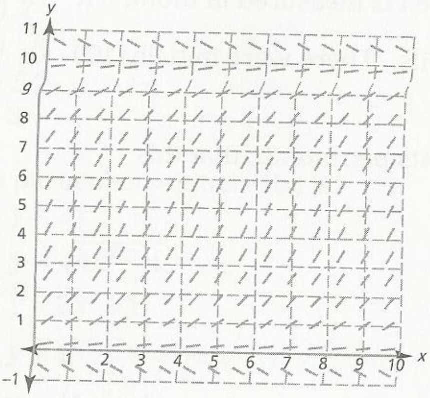
A population of rabbits in a certain habitat grows according to the differential equation $\frac{dy}{dt} = y\left(1 - \frac{1}{10}y\right)$ where t is measured in months ($t \geq 0$) and y is measured in hundreds of rabbits. There were initially 100 rabbits in this habitat; that is, $y(0) = 1$.

*8. What is the fastest growth rate, in rabbits per month, that this population exhibits?

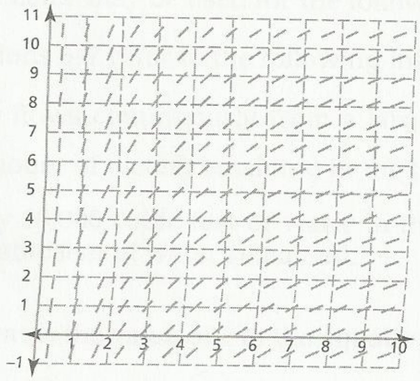
(A) 50
(B) 100
(C) 200
(D) 250
(E) 500

*9. Which of the following slope fields represents an approximate general solution to the given differential equation?

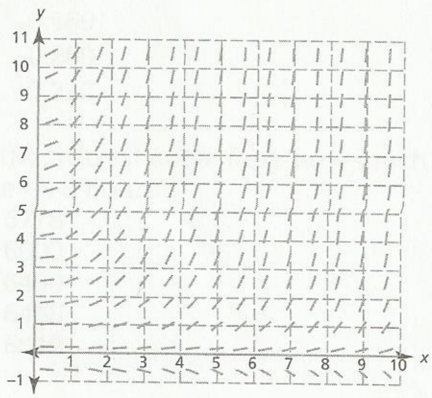
(A)



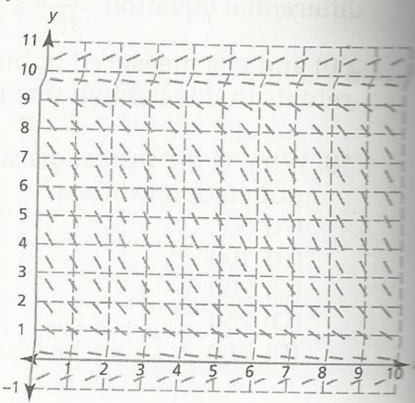
(B)



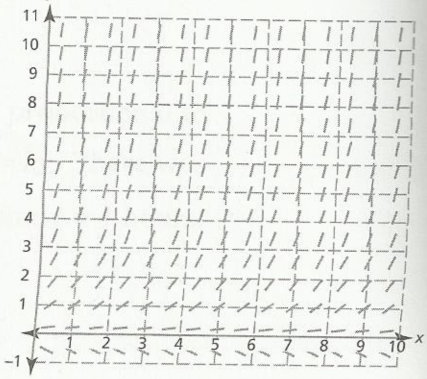
(C)



(D)



(E)



*10. Estimates of $y(t)$ can be produced using Euler's Method with step size $\Delta t = 1$. To the nearest rabbit, the estimate for $y(2)$ is

(A) 281
(B) 300
(C) 344
(D) 379
(E) 500

11. Water is being pumped continuously into a tank at a rate that is inversely proportional to the amount of water in the tank; that is, $\frac{dy}{dt} = \frac{k}{y}$, where y is the number of gallons of water in the tank after t minutes $(t \geq 0)$. Initially there were 5 gallons of water in the tank, and after 3 minutes there were 7 gallons. How many gallons of water were in the tank at time 18 minutes?

(A) $\sqrt{61}$
(B) $\sqrt{97}$
(C) 13
(D) $\sqrt{201}$
(E) 17

A calculator may not be used for the following questions.

Questions 12 and 13 refer to the following information:

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2} y \cos(x)$, for which the solution is $y = f(x)$. Let $f(0) = 2$.

12. Which of the following statements about the graph of $f(x)$ are true?
I. $f(x)$ has a vertical tangent when $y = 0$.
II. $f(x)$ has a horizontal tangent when $x = \frac{\pi}{2}$.
III. The slope of $f(x)$ at the point $(0,2)$ is 1.

(A) I only
(B) II only
(C) I and II only
(D) II and III only
(E) I, II, and III

13. The particular solution is

(A) $f(x) = x + 2$
(B) $f(x) = 2e^{-\frac{1}{2}\sin(x)}$
(C) $f(x) = \sqrt{\sin(x) + 4}$
(D) $f(x) = e^{\frac{1}{2}\sin(x)}$
(E) $f(x) = 2e^{\frac{1}{2}\sin(x)}$

Questions 14 and 15 refer to the following information:

Consider the differential equation $\frac{dy}{dx} = x + 2y$, for which the solution is $g(x)$.

14. Which of the following statements is true about the particular solution that contains $(0,-1)$?
(A) $g(x)$ is increasing and concave up.
(B) $g(x)$ is increasing and concave down.
(C) $g(x)$ is decreasing and concave up.
(D) $g(x)$ is decreasing and concave down.
(E) $g(x)$ is decreasing and linear.

*15. Let $y(x)$ be the particular solution that contains $(0,1)$. Using Euler's Method with step size $\Delta x = \frac{1}{2}$, what is the estimate for $y\left(\frac{1}{2}\right)$?

(A) $-\frac{1}{4}$
(B) $\frac{3}{4}$
(C) 1
(D) $\frac{3}{2}$
(E) 2

FREE-RESPONSE QUESTION

A calculator may not be used on this question.

1. A differentiable function $f(x)$ is defined such that, for all values of x in its domain, $f(x) = 3 + \int_8^{x^3} f\left(\sqrt[3]{t}\right)\, dt$.
   a. What is the domain of $f(x)$?
   b. For what value(s) of x is $f(x) = 3$?
   c. Show that $f'(x) = 3x^2 f(x)$.
   d. Solve the differential equation in (c) to find $f(x)$ in terms of x only.

Answers

Multiple Choice

1. **(D)** Separating variables,
$\int \frac{dy}{y^2} = \int \sin x\, dx \Rightarrow -\frac{1}{y} = -\cos x + C \Rightarrow y = \frac{1}{\cos x} + C = \sec x + C.$
(*Calculus* 8th ed. pages 421–428 / 9th ed. pages 423–430)

2. **(B)** Separating variables, $\int e^y dy = \int 2x\, dx \Rightarrow e^y = x^2 + C$. Using the initial value (1, 2), $e^2 = 1 + C \Rightarrow C = e^2 - 1$. Therefore, $e^y = x^2 + e^2 - 1 \Rightarrow y = \ln(x^2 + e^2 - 1)$. (*Calculus* 8th ed. pages 421–428 / 9th ed. pages 423–430)

3. **(B)** The first clear feature of the correct slope field is that all the slopes must be 0 on the y-axis (where $x = 0$). This eliminates choices (A), (C), and (D). The appearance of a line (approximately $y = 2x$) in the slope field in (B) happens because, wherever y is twice x, the slope is 2. The sinusoidal nature of (E) should suggest a sine or cosine function in the differential equation, which eliminates (E). So (B) is correct (*Calculus* 8th ed. pages 404–407 / 8th ed. pages 406–409).

4. **(B)** The formula for Euler's Method with a given expression for $\frac{dy}{dx}$, a step size Δx, and an initial condition (x_0, y_0) is
$$y_{n+1} = y_n + \left.\frac{dy}{dx}\right|_{(x_n, y_n)} \cdot \Delta x.$$
Using this,
$$x_0 = 0 \qquad y_0 = 1 \qquad \left.\frac{dy}{dx}\right|_{(0,1)} = 0$$
$$x_1 = \frac{1}{2} \qquad y_1 = 1 + 0 \cdot \frac{1}{2} = 1 \qquad \left.\frac{dy}{dx}\right|_{\left(\frac{1}{2},1\right)} = 2$$
$$x_2 = 1 \qquad y_2 = 1 + 2 \cdot \frac{1}{2} = 2$$
Therefore $y(1) \approx 2$. (*Calculus* 8th ed. pages 404–409 / 9th ed. pages 406–411)

5. **(E)** Separating variables,
$\int y\, dy = \int 4x\, dx \Rightarrow \frac{1}{2}y^2 = 2x^2 + C \Rightarrow y^2 = 4x^2 + C$. Using the initial value (0, 1), $1 = 0 + C \Rightarrow C = 1$. Therefore $y = \sqrt{4x^2 + 1}$.
(*Calculus* 8th ed. pages 421–428 / 9th ed. pages 423–430)

6. **(B)** Separating variables,
$\int \frac{dy}{y} = \int k\, dt \Rightarrow \ln|y| = kt + C \Rightarrow y = Ce^{kt}$,

$y(0) = 10000 \Rightarrow C = 10000$, so $y = 10000e^{kt}$,

$y(4) = 8000 \Rightarrow 8000 = 10000e^{k \cdot 4} \Rightarrow$

$.8 = e^{4k} \Rightarrow k = \dfrac{\ln(.8)}{4} \approx -.056$. (*Calculus* 8th ed. pages 413–417 / 9th ed. pages 415–419)

7. (C) Using the correct equation from Question 6, $y = e^{\frac{1}{4}\ln(.8)t} \Rightarrow y(8) = 6400$. (*Calculus* 8th ed. pages 413–417 / 9th ed. pages 413–419)

8. (D) The question asks for the maximum value of $\dfrac{dy}{dt}$. So the derivative of $\dfrac{dy}{dt}$ must be set equal to zero: $\dfrac{d}{dy}\left[y - \dfrac{1}{10}y^2\right]$ $\Rightarrow 1 - \dfrac{1}{5}y = 0 \Rightarrow y = 5$. Since $1 - \dfrac{1}{5}y$ changes from positive to negative at $y = 5$, a maximum occurs at $y = 5$. $\left.\dfrac{dy}{dt}\right|_{y=5} = 5 - \dfrac{1}{10}(25) = 2.5$, so the maximum growth rate is 250 rabbits per month when the number of rabbits is 500. (*Note:* You may now realize that the logistic equation increases most rapidly when it is halfway to its carrying capacity and that the carrying capacity for this particular equation is 10. Therefore, you can quickly conclude that $y = 5$.) (*Calculus* 8th ed. pages 164–168 / 9th ed. pages 164–168)

9. (A) The given differential equation should be recognized as having the form of logistic growth. Solutions to this should increase slowly from the initial point, then increase faster, and then finally more slowly as the population size nears its carrying capacity. (A) is the only choice that exhibits these characteristics. (*Calculus* 8th ed. pages 404–408 / 9th ed. pages 406–410)

10. (C) The formula for Euler's Method with a given expression for $\dfrac{dy}{dt}$, a step size Δt, and an initial condition (t_0, y_0) is

$$y_{n+1} = y_n + \left.\frac{dy}{dt}\right|_{(t_n,\, y_n)} \cdot \Delta t.$$

Using this,

$t_0 = 0 \quad y_0 = 1 \quad \left.\dfrac{dy}{dt}\right|_{(0,1)} = .9$

$t_1 = 1 \quad y_1 = 1 + 0.9 \cdot 1 = 1.9 \quad \left.\dfrac{dy}{dt}\right|_{(1,1.9)} = 1.9\left[1 - \dfrac{1}{10}(1.9)\right] = 1.539$

$t_2 = 2 \quad y_2 = 1.9 + 1.539 \cdot 1 = 3.439$.

Therefore $y(2) \approx 3.439$, so the answer is 344 rabbits. (*Calculus* 8th ed. pages 404–408 / 9th ed. pages 406–410)

11. **(C)** $\frac{dy}{dt}=\frac{k}{y}$. Separating variables,

$$\int y\,dy=\int k\,dt \Rightarrow \frac{1}{2}y^2=kt+C \Rightarrow y^2=2kt+C \Rightarrow y=\sqrt{2kt+C}.$$

Using the initial condition $(0,5)$, $5=\sqrt{C}\Rightarrow C=25$. Using $(3,7)$, $7=\sqrt{6k+25}\Rightarrow k=4\Rightarrow y=\sqrt{8t+25}$.

Thus $y(18)=\sqrt{144+25}=13$. (*Calculus,* 8th ed. pages 413–417/9th ed. pages 415–419)

12. **(D)** Statement I is false because $\frac{dy}{dx}=0$ *when* $y=0$. $\frac{dy}{dx}$ would need to be undefined in order for there to be a vertical tangent *when* $y=0$. Statement II is true because $\frac{dy}{dx}=0$ *when* $x=\frac{\pi}{2}$.

Statement III is true because $\left.\frac{dy}{dx}\right|_{(0,2)}=.5(2)\cos(0)=1$ (*Calculus* 8th ed. pages 413 - 418 / 9th ed. pages 415 - 420)

13. **(E)** Separating variables,

$$\int\frac{dy}{y}=\int\frac{1}{2}\cos(x)dx \Rightarrow \ln|y|=\frac{1}{2}\sin(x)+C \Rightarrow y=Ce^{\frac{1}{2}\sin(x)}.$$

Using the initial value $(0,2)$, $2=Ce^0\Rightarrow C=2$. Therefore $f(x)=2e^{\frac{1}{2}\sin(x)}$. (*Calculus,* 8th ed. pages 413–417/9th ed. pages 415–419)

14. **(D)** $\left.\frac{dy}{dx}\right|_{(0,-1)}=0-2=-2<0\Rightarrow g(x)$ is decreasing. To determine concavity, the sign of $\frac{d^2y}{dx^2}$ must be established. $\left.\frac{d^2y}{dx^2}\right|_{(0,-1)}=1+2\left.\frac{dy}{dx}\right|_{(0,-1)}=1+2(-2)=-3<0\Rightarrow g(x)$ is concave down. (*Calculus,* 8th ed. pages 413–418/9th ed. pages 415–420)

15. **(E)** The formula for Euler's Method with a given expression for $\frac{dy}{dx}$, a step size Δx, and an initial condition (x_0,y_0) is $y_{n+1}=y_n+\left.\frac{dy}{dx}\right|_{(x_n,y_n)}\cdot\Delta x.$

Using this,

$x_0=0$ $\quad y_0=1$ $\quad \left.\frac{dy}{dx}\right|_{(0,1)}=2$

$x_1=\frac{1}{2}$ $\quad y_1=1+2\cdot\frac{1}{2}=2$

Therefore $y\left(\frac{1}{2}\right) \approx 2$. (*Calculus,* 8th ed. page 408/9th ed. page 410)

FREE RESPONSE

	Solution	Possible points
a.	$\sqrt[3]{t}$ is defined for all real numbers, so x^3 can be any real number; therefore x can be any real number.	1: answer
b.	$f(x) = 3$ when $x^3 = 8$; therefore $x = 2$ because the integral from 8 to 8 = 0.	1: answer
c.	$f'(x) = f\left(\sqrt[3]{x^3}\right) \cdot 3x^2$, so $f'(x) = 3x^2 f(x)$.	2: { 1: argument of $f(x)$; 1: Chain Rule }
d.	Rewrite $f'(x) = 3x^2 f(x)$ as $\frac{dy}{dx} = 3x^2 y$. Separating variables, $\int \frac{dy}{y} = \int 3x^2\, dx$ $\ln\lvert y\rvert = x^3 + C$ $y = e^{x^3+C} = Ce^{x^3}$. Using $f(2) = 3$ (from answer b), we get $3 = Ce^8 \Rightarrow C = 3e^{-8}$ Therefore $y = 3e^{-8}e^{x^3}$, hence $f(x) = 3e^{x^3-8}$.	5: { 1: separation of variables; 2: correct antiderivatives; 1: includes constant of integration; 1: solves correctly for constant }

(a), (b), and (c) (*Calculus* 8th ed. pages 288–290 / 9th ed. pages 238–290)
(d) (*Calculus* 8th ed. page 421 / 9th ed. page 423)