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Quiz#1

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You have 20 minutes to finish the following 3 problems.

1. (5 points) Solve the following I.V.P.

$$\frac{y'}{t} + \frac{y}{t^2} = 5, \quad y(1) = 1.$$

(1) Standard form: (multiply by t)

$$y' + \frac{y}{t} = 5t.$$

Note that by uniqueness and existence theorem, since $p(t) = \frac{1}{t}$ and $g(t) = 5$ are continuous everywhere except in $t=0$ for $p(t)$, and $t=1$, we will have a unique solution on $(0, \infty)$. Solve the eq by integrating factor $u(t) = e^{\int \frac{1}{t} dt} = e^{\ln(t)} = t$.

$\Rightarrow u(t) = t$.

(2) multiply both sides by $u(t)$: $t \left[y' + \frac{y}{t} \right] = 5t^2$

(3) By product rule: $\frac{d}{dt} [ty] = 5t^2$

(4) Integrate both sides $\int \frac{d}{dt} [ty] dt = \int 5t^2 dt$

(5) solve:

$$ty = \frac{5}{3}t^3 + C \Rightarrow \boxed{y = \frac{5}{3}t^2 + \frac{C}{t}}$$

this is the general sol.
finally, solve for C :

the particular solution is:

$$y(1) = 1 = \frac{5}{3} + C \Rightarrow C = 1 - \frac{5}{3} \Rightarrow \boxed{C = -\frac{2}{3}}$$

$$\boxed{y(t) = \frac{5}{3}t^2 - \frac{2}{3t}}$$

2. (5 points) State the existence and uniqueness theorem for linear first order differential equations and determine where the solution of the given I.V.P. is certain to exist and be unique.

$$(t-4)y' + \ln(t-1)y = \frac{t^2}{t-6}, \quad y(2) = 6.$$

The existence and uniqueness theorem

Consider the I.V.P. (*)
$$\begin{cases} y' + p(t)y = q(t) \\ y(t_0) = y_0 \end{cases}$$

If $p(t)$ and $q(t)$ are continuous over an interval $\alpha < t < \beta$
AND $t_0 \in (\alpha, \beta)$.


THEN there exists a unique solution $\phi(t)$ to (*).

Consider the equation above. To use theorem, first write in standard form (multiplying by $(t-4)^{-1}$):

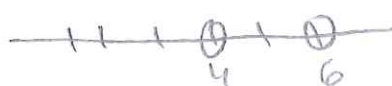
$$y' + \frac{\ln(t-1)}{t-4} y = \frac{t^2}{(t-4)(t-6)}$$

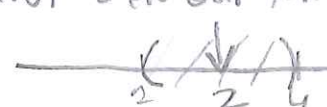
Now, let $p(t) = \frac{\ln(t-1)}{t-4}$ AND $q(t) = \frac{t^2}{(t-4)(t-6)}$ the function

$p(t)$ is continuous if $t > 1$ (since $\ln(t-1)$ cannot be negative) AND $t \neq 4$



$q(t)$ is continuous if $t \neq 4$ AND $t \neq 6$.



Now, by U.E.T., given that our initial condition is $t_0 = 2$, we will have a solution (unique)  if $t \in (1, 4)$

3. (5 points) Find the general solution of the following differential equation.

$$y' + 2y = 7ty^2.$$

this is a 1st O.D.E, non linear, fitting the Bernoulli CASE when $n=2$. It is already in standard form.

We make the change: $u = y^{1-n} = y^{-1}$

$$u' = -1 y^{-2} y' \Rightarrow y' y^{-2} = -u'$$

Multiply the original eq. by y^{-2} :

$$y' y^{-2} + 2y^{-1} = 7t$$

Make the change

$$-u' + 2u = 7t \quad \checkmark \quad \text{Now, this is a 1st O.D.E. linear. Solve by integrating factor.}$$

(1) Write in standard form: $u' - 2u = -7t$

(2) integrating factor: $e^{\int -2 dt} = e^{-2t} \quad \checkmark$ (b) change back to y:

$$(3) e^{-2t} [u' - 2u] = -7te^{-2t}$$

$$(4) \int \frac{d}{dt} [e^{-2t} u] = \int -7te^{-2t} dt \quad \checkmark$$

$$(5) e^{-2t} u = -7 \int_0^t s e^{-2s} ds + C \quad \times$$

$$u = \left(-7 \int_0^t s e^{-2s} ds + C \right) / e^{-2t}$$

$$\Rightarrow y(t) = \frac{e^{-2t}}{t - 7 \int_0^t s e^{-2s} ds + C}$$

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