M343 Homework 4

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Section 3.2

3. Consider the pair of functions: e^{-2t}, te^{-2t}. Then,

$$\begin{aligned} W(t) &= e^{-2t}(te^{-2t})' - te^{-2t}(e^{-2t})' \\ &= e^{-2t}(e^{-2t} - 2te^{-2t}) + 2te^{-4t} \\ &= e^{-4t} - 2te^{-4t} + 2te^{-4t} \\ &= e^{-4t} \end{aligned}$$

Hence, the Wronskian of the given pair of functions is $W(t) = e^{-4t}$.

6. Consider the pair of functions: $cos^2(\theta), 1 + cos(2\theta)$. Note that, $1 + cos(2\theta) = 2cos^2(\theta)$ Then,

$$\begin{aligned} W(\theta) &= (cos^2(\theta))(2cos^2(\theta))' - (cos^2(\theta)'(2cos^2(\theta)) \\ &= (cos^2(\theta))2(-2sin(\theta)cos(\theta)) - (-2sin(\theta)cos(\theta))(2cos^2(\theta)) \\ &= -4sin(\theta)cos^3(\theta) + 4sin(\theta)cos^3(\theta) \\ &= 0 \end{aligned}$$

Hence, the Wronskian of the given pair of functions is $W(\theta) = 0$.

10. $y'' + (cos(t))y' + 3(ln(|t|))y = 0, \quad y(2) = 3, \quad y'(2) = 1.$ This equation is already in standard form.

Applying Theorem 3.2.1 (E.U.T), we analyze continuity of the functions $p(t) = cos(t)$ (continuos everywhere), $q(t) = 3(ln(|t|))$ (continuos everywhere except at $t = 0$) and $g(t) = 0$ (continuos everywhere).

Hence, these functions are continuous when either $-\infty < t < 0$ or $0 < t < \infty$. But, $t_0 = 2 \in (0, \infty)$ and hence, there exists a unique solution in this interval.

11. $(x - 3)y'' + xy' + (ln(|x|))y = 0, \quad y(1) = 0, \quad y'(1) = 1.$ This equation is not in standard form. Multiplying by $\dfrac{1}{x-3}$ we can convert this equation to standard form: $y'' + \dfrac{x}{x-3} + \dfrac{ln(|x|)}{x-3} = 0$

Applying Theorem 3.2.1 (E.U.T), we analyze continuity of the functions $p(t) = \dfrac{x}{x-3}$ (continuos everywhere except at $x = 3$), $q(t) = \dfrac{ln(|x|)}{x-3}$ (continuos everywhere except at $t = 0$ and $t = 3$) and $g(t) = 0$ (continuos everywhere).

Hence, these functions are continuous when either $-\infty < t < 0$ or $0 < t < 3$ or $3 < t < \infty$. But, $t_0 = 1 \in (0, 3)$ and hence, there exists a unique solution in this interval.

13. Consider the following linear, homogeneous, 2nd O.D.E: $\quad t^2y'' - 2y = 0, \quad t > 0.$

<u>Claim:</u> $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are solutions to the O.D.E. <u>Proof:</u> They each have to satisfy the equation.

Note that $y_1'' = 2$ and $y_2'' = \dfrac{2}{t^3}$

For y_1 we have: $t^2y_1'' - 2y_1 = t^2(2) - 2t^2 = 0$. So, y_1 is a solution.
For y_2 we have: $t^2y_2'' - 2y_2 = \dfrac{2}{t} - \dfrac{2}{t} = 0$. So, y_2 is a solution.

<u>Claim:</u> $y = c_1t^2 + c_2t^{-1}$ is also a solution.
<u>Proof:</u> $t^2y'' - 2y = t^2(2c_1 + \dfrac{2c_2}{t^3}) - 2(c_1t^2 + c_2t^{-1}) = 2c_1t^2 + \dfrac{2c_2}{t} - 2c_1t^2 - \dfrac{2c_2}{t} = 0$, so y is also a solution for any c_1, c_2.

17. Let $W(t) = 3e^{4t}$ and $f(t) = e^{2t}$. We want to find $g(t)$. Applying the definition of the Wronskian:

$$\begin{aligned} W(t) &= f \cdot g' - f' \cdot g \\ &= e^{2t} \cdot g' - 2e^{2t} \cdot g \\ 3e^{4t} &= e^{2t} \cdot g' - 2e^{2t} \cdot g \\ 3e^{4t} &= e^{2t}(g' - 2g) \iff \\ 3e^{2t} &= g' - 2g \end{aligned}$$

This final equation is a linear, 1st O.D.E. We solve this by integrating factor $\mu(t) = e^{\int -2} = e^{-2t}$.:

$$\frac{d}{dt}[e^{-2t}g] = 3e^{2t} \cdot e^{-2t} \implies \text{integrating both sides } \; e^{-2t}g = \int 3dt = 3t + C$$

So, the function g is given by $g(t) = 3te^{2t} + Ce^{2t}$

22.

$$y'' + y' - 2y = 0, \qquad t_0 = 0$$

The characteristic equation is $r^2 + r - 2 = 0 \iff (r-1)(r+2) = 0$. The solution is given by:

$$y(t) = C_1e^t + C_2e^{-2t}$$

By Theorem 3.2.5, let y_1 be the solution that satisfies $y_1(t_0) = 1, \;\; y_1'(t_0) = 0$. Then:

$$\begin{cases} y_1(t_0) = y_1(0) = 1 = C_1 + C_2 & \implies 1 = 3C_2 \implies C_2 = \dfrac{1}{3} \\ \\ y_1'(t_0) = y_1'(0) = 0 = C_1 - 2C_2 & \implies C_1 = 2C_2 \implies C_1 = \dfrac{2}{3} \end{cases}$$

The particular solution y_1 is given by $y_1(t) = \dfrac{2}{3}e^t + \dfrac{1}{3}e^{-2t}$

Likewise, by Theorem 3.2.5, let y_2 be the solution that satisfies $y_2(t_0) = 1, \;\; y_2'(t_0) = 0$. Then:

$$\begin{cases} y_2(t_0) = y_2(0) = 0 = C_1 + C_2 & \implies C_1 = -C_2 \implies C_1 = \dfrac{1}{3} \\ \\ y_2'(t_0) = y_2'(0) = 1 = C_1 - 2C_2 & \implies 1 = -3C_2 \implies C_2 = -\dfrac{1}{3} \end{cases}$$

The particular solution y_2 is given by $y_2(t) = \dfrac{1}{3}e^t - \dfrac{1}{3}e^{-2t}$

By Theorem 3.2.5, y_1 and y_2 form a fundamental set of solutions.

23.

$$y'' + 4y' + 3y = 0, \qquad t_0 = 1$$

The characteristic equation is $r^2 + 4r + 3 = 0 \iff (r+1)(r+3) = 0$. The solution is given by:

$$y(t) = C_1e^{-t} + C_2e^{-3t}$$

By Theorem 3.2.5, let y_1 be the solution that satisfies $y_1(t_0) = 1, \;\; y_1'(t_0) = 0$. Then:

$$\begin{cases} y_1(t_0) = y_1(1) = 1 = C_1e^{-1} + C_2e^{-3} & \implies -\dfrac{3}{2} = -C_1e^{-1} \implies C_1 = \dfrac{3e}{2} \\ \\ y_1'(t_0) = y_1'(1) = 0 = -C_1e^{-1} - 3C_2e^{-3} & \implies 1 = -2C_2e^{-3} \implies C_2 = -\dfrac{e^3}{2} \end{cases}$$

The particular solution y_1 is given by $y_1(t) = \dfrac{3e}{2}e^{-t} - \dfrac{e^3}{2}e^{-3t} \iff y_1(t) = \dfrac{3e^{1-t}}{2} - \dfrac{e^{3-3t}}{2}$

Likewise, by Theorem 3.2.5, let y_2 be the solution that satisfies $y_2(t_0) = 1, \;\; y_2'(t_0) = 0$. Then:

$$\begin{cases} y_2(t_0) = y_2(1) = 0 = C_1e^{-1} + C_2e^{-3} & \implies C_1e^{-1} = \dfrac{1}{2} \implies C_1 = \dfrac{e}{2} \\ \\ y_2'(t_0) = y_2'(1) = 1 = -C_1e^{-1} - 3C_2e^{-3} & \implies 1 = -2C_2e^{-3} \implies C_2 = -\dfrac{e^3}{2} \end{cases}$$

The particular solution y_2 is given by $y_2(t) = \dfrac{e^{1-t}}{2} - \dfrac{e^{3-3t}}{2}$

By Theorem 3.2.5, y_1 and y_2 form a fundamental set of solutions.

Section 3.3

9. Consider the following homogeneous, linear, 2nd O.D.E with constant coefficients:

$$y'' + 2y' - 8y = 0$$

The characteristic equation is $r^2 + 2r - 8 = 0 \iff (r+4)(r-2) = 0$. Hence, the general solution is given by:

$$y(t) = C_1 e^{-4t} + C_2 e^{2t}$$

16. Consider the following homogeneous, linear, 2nd O.D.E with constant coefficients:

$$y'' + 4y' + 6.25y = 0$$

The characteristic equation is $r^2+4r-6.25 = 0$. Solving via quadratic formula: $r = \dfrac{-4 \pm \sqrt{16-25}}{2 \cdot 1} = -2 \pm \dfrac{3}{2}i$. We have two complex roots: $r_1 = -2 + \frac{3}{2}i$ and $r_2 = -2 + \frac{3}{2}i$. The solutions are:

$$y_1(t) = e^{(-2+\frac{3}{2}i)t} = e^{-2t}[cos(\frac{3}{2}t) + isin(\frac{3}{2}t)]$$

$$y_2(t) = e^{(-2-\frac{3}{2}i)t} = e^{-2t}[cos(-\frac{3}{2}t) + isin(-\frac{3}{2}t)] = \text{ trig. identities } = e^{-2t}[cos(\frac{3}{2}t) - isin(\frac{3}{2}t)]$$

To express the solution in terms of real-valued functions, we use the fact that the solutions are closed under addition:

$$y_1(t) + y_2(t) = 2e^{-2t}cos(\frac{3}{2}t)$$

$$y_1(t) - y_2(t) = 2ie^{-2t}sin(\frac{3}{2}t)$$

We can drop the constants 2 and $2i$ to obtain the general solution:

$$y(t) = C_1 e^{-2t}cos(\frac{3}{2}t) + C_2 e^{-2t}sin(\frac{3}{2}t)$$

19. Consider the I.V.P: $y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$. This is a homogeneous, linear, 2nd O.D.E with constant coefficients. To solve it we find the characteristic equation:

$$r^2 - 2r + 5 = 0 \iff r = \frac{2 \pm \sqrt{4-20}}{2 \cdot 1} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

We have two complex roots: $r_1 = 1 + 2i$ and $r_2 = 1 - 2i$. The solutions are:

$$y_1(t) = e^{(1+2i)t} = e^t[cos(2t) + isin(2t)]$$

$$y_2(t) = e^{(1-2i)t} = e^t[cos(-2t) + isin(-2t)] = e^t[cos(2t) - isin(2t)]$$

To express the solution in terms of real-valued functions, we use the fact that the solutions are closed under addition:

$$y_1(t) + y_2(t) = 2e^t cos(2t)$$

$$y_1(t) - y_2(t) = 2ie^t sin(2t)$$

We can drop the constants 2 and $2i$ to obtain the general solution:

$$y(t) = C_1 e^t cos(2t) + C_2 e^t sin(2t)$$

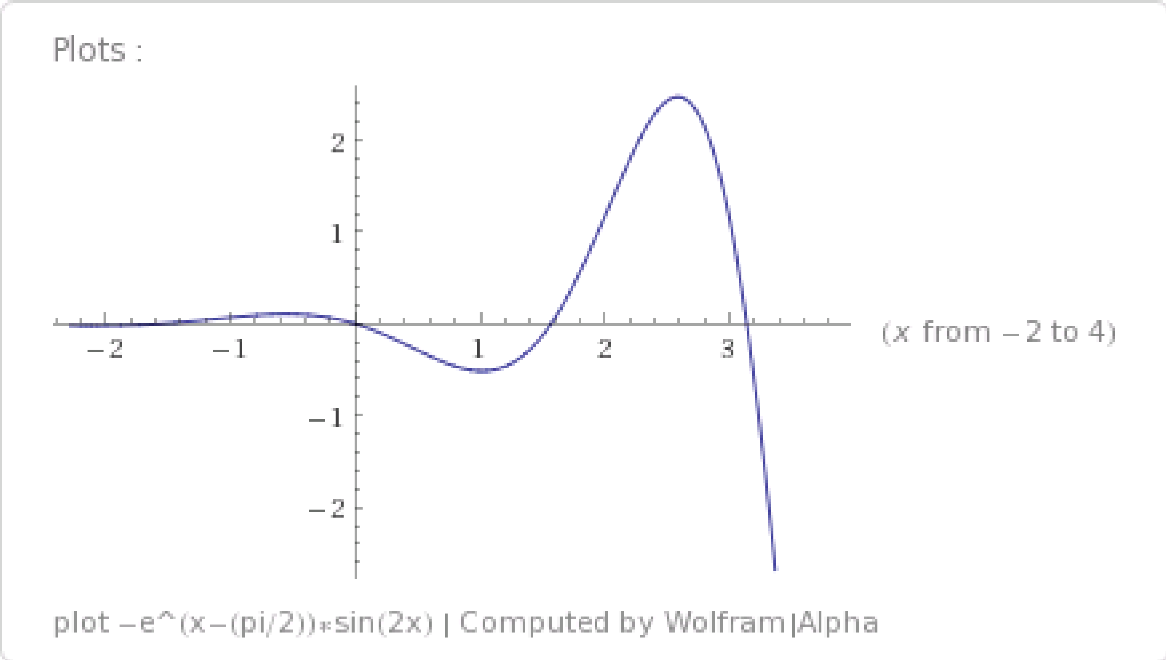
Finally, we solve for the initial conditions:

$$\begin{cases} y(\pi/2) = 0 = C_1 e^{\pi/2}cos(\pi) + C_2 e^{\pi/2}sin(\pi) = -e^{\pi/2}C_1 \Longrightarrow C_1 = 0 \\ \\ y'(\pi/2) = 2 = \text{ (we already know that } C_1 = 0 \text{ ... }) = C_2[e^{\pi/2}sin(\pi) + e^{\pi/2}cos(\pi)\frac{\pi}{2} = -2C_2 e^{\pi/2}C_2 \Longrightarrow \frac{-1}{e^{\pi/2}} \end{cases}$$

So, the particular solution for the I.V.P:

$$y(t) = \frac{-1}{e^{\pi/2}}e^t sin(2t) \iff y(t) = -e^{t-\pi/2}sin(2t)$$

Graph of the solution:



Also, $\lim_{t\to\infty} y(t) = -\infty$

21. Consider the I.V.P: $y'' + y' + 1.25y = 0, \quad y(0) = 3, \quad y'(0) = 1$. This is a homogeneous, linear, 2nd O.D.E with constant coefficients. To solve it we find the characteristic equation:

$$r^2 + r + 1.25 = 0 \iff r = \frac{-1 \pm \sqrt{1-5}}{2 \cdot 1} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i$$

We have two complex roots: $r_1 = -\frac{1}{2} + i$ and $r_2 = -\frac{1}{2} - i$. The solutions are:

$$y_1(t) = e^{(i-\frac{1}{2})t} = e^{-\frac{t}{2}}[cos(t) + isin(t)]$$

$$y_2(t) = e^{(-i-\frac{1}{2})t} = e^{-\frac{t}{2}}[cos(-t) + isin(-t)] = e^{-\frac{t}{2}}[cos(t) - isin(t)]$$

To express the solution in terms of real-valued functions, we use the fact that the solutions are closed under addition:

$$y_1(t) + y_2(t) = 2e^{-\frac{t}{2}}cos(t)$$

$$y_1(t) - y_2(t) = 2ie^{-\frac{t}{2}}sin(t)$$

We can drop the constants 2 and $2i$ to obtain the general solution:

$$y(t) = C_1e^{-\frac{t}{2}}cos(t) + C_2e^{-\frac{t}{2}}sin(t)$$

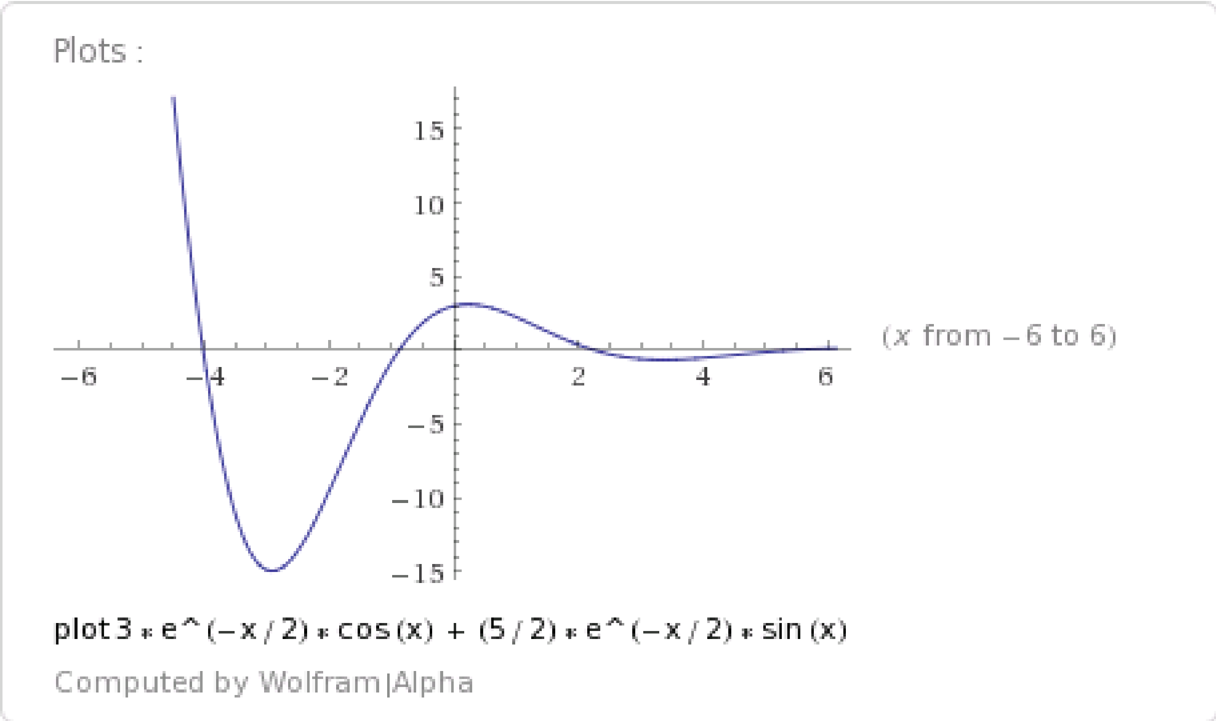
Finally, we solve for the initial conditions:

$$\begin{cases} y(0) = 3 = C_1e^0cos(0) + C_2e^0sin(0) \Longrightarrow C_1 = 3 \\ \\ y'(0) = 1 = C_1[-\frac{1}{2}e^0cos(0) - e^0sin(0)] + C_2[-\frac{1}{2}e^0sin(0) + e^0cos(0)] = -\frac{1}{2}C_1 + C_2 \Longrightarrow C_2 = \frac{5}{2} \end{cases}$$

So, the particular solution for the I.V.P:

$$y(t) = 3e^{-\frac{t}{2}}cos(t) + \frac{5}{2}e^{-\frac{t}{2}}sin(t)$$

Graph of the solution:



Also, $\lim_{t\to\infty} y(t) = 0$