(1) Exercise 6.2.23: Let B be the unit ball. Evaluate 100

$$\iiint_B \frac{dx\,dy\,dz}{\sqrt{2+x^2+y^2+z^2}}$$, by making the appropiate change of variables.

Solution: the appropiate change is to spherical coordinates.

$$\iiint_B \frac{dx\,dy\,dz}{\sqrt{2+x^2+y^2+z^2}} = \int_0^{\pi}\int_0^{2\pi}\int_0^1 \frac{\rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi}{\sqrt{2+\rho^2}} = 2\pi \int_0^{\pi} \sin\varphi\, d\varphi \int_0^1 \frac{\rho^2}{\sqrt{2+\rho^2}}\, d\rho$$

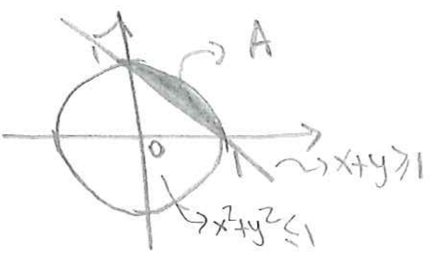
$$= 4\pi\left[\frac{\rho\sqrt{\rho^2+2}}{2} - \ln\left(\rho+\sqrt{\rho^2+2}\right)\right]_0^1$$ (integral from back of the book).

$$= \boxed{4\pi\left[\frac{\sqrt{3}}{2} - \ln(1+\sqrt{3}) + \ln(\sqrt{2})\right]}$$

(2) Exercise 6.2.24. Evaluate $\iint_A [1/(x^2+y^2)^2]\,dx\,dy$, where A is determined by the conditions:

$$x^2+y^2 \leq 1 \quad \text{and} \quad x+y \geq 1.$$

Solution: First, let us plot the region A:



Let us change to polar coordinates:

$x^2+y^2 \leq 1 \Rightarrow r^2 \leq 1 \Rightarrow r \leq 1, (r \geq 0)$.

$x+y \geq 1 \Rightarrow y \geq 1-x \Rightarrow r\sin\theta \geq 1 - r\cos\theta$

$\Rightarrow r(\sin\theta+\cos\theta) \geq 1 \Rightarrow r \geq \frac{1}{\sin\theta+\cos\theta}$, which is never zero.

So the region in polar coordinates is $\{(r,\theta) : \frac{1}{\sin\theta+\cos\theta} \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\}$

$$\iint_A \frac{1}{(x^2+y^2)^2}\, dA = \int_0^{\pi/2}\int_{\frac{1}{\sin\theta+\cos\theta}}^1 \frac{r\, dr\, d\theta}{r^4} = \int_0^{\pi/2}\int_{1/\sin\theta+\cos\theta}^1 \frac{dr\,d\theta}{r^3} = \int_0^{\pi/2}\left[\frac{-1}{2r^2}\right]_{\frac{1}{\sin\theta+\cos\theta}}^1 d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} 1-(\sin\theta+\cos\theta)^2\, d\theta = -\frac{1}{2}\int_0^{\pi/2} 1-(\sin^2\theta + 2\sin\theta\cos\theta+\cos^2\theta)\, d\theta$$

$$= \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \left[-\frac{1}{2}\cos^2(\theta)\right]_0^{\frac{\pi}{2}} = -\frac{1}{2}\cos^2\left(\frac{\pi}{2}\right) + \frac{1}{2}\cos^2(0) = \boxed{\frac{1}{2}}$$

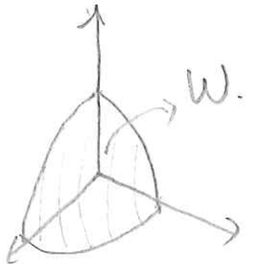
(3) Exercise 6.2.26. Use spherical coordinates to evaluate.

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} \frac{\sqrt{x^2+y^2+z^2}}{1+[x^2+y^2+z^2]^2}\, dz\, dy\, dx$$

Solution: the volume W we are integrating over is

$$W = \{(x,y,z):\ 0 \le x \le 3;\ 0 \le y \le \sqrt{9-x^2};\ 0 \le z \le \sqrt{9-x^2-y^2}\}.$$

this volume is the volume in the first octant enclosed by the xy-plane and the sphere of radius 3.



this region in spherical coordinates is:

$$W_\rho = \{(\rho,\varphi,\theta): 0 \le \rho \le 3;\ 0 \le \varphi \le \tfrac{\pi}{2};\ 0 \le \theta \le \tfrac{\pi}{2}\}$$

the integral becomes:

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \frac{\sqrt{\rho^2}}{1+[\rho^2]^2}\rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi. = \frac{\pi}{2}\int_0^{\pi/2} \sin\varphi\, d\varphi \int_0^3 \frac{\rho^3}{1+\rho^4} d\rho$$

$$= \frac{\pi}{2}\left[-\cos\varphi\right]_0^{\pi/2} \int_0^3 \frac{\rho^3}{1+\rho^4} d\rho = \frac{\pi}{2}\int_0^3 \frac{\rho^3}{1+\rho^4} d\rho \ ;$$ change of variables $1+\rho^4 = u \Rightarrow du = 4\rho^3 d\rho \Rightarrow \frac{du}{4} = \rho^3 d\rho$

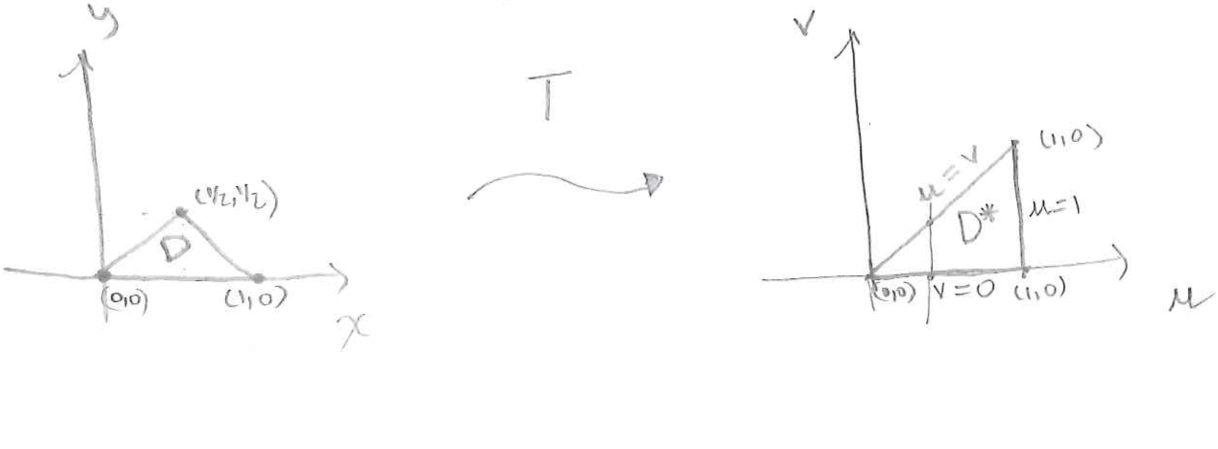
$$\rightsquigarrow \frac{\pi}{2}\int_{\square}^{\square} \frac{du}{4(u)} = \frac{\pi}{8}\int_{\square}^{\square} \frac{du}{u} = \frac{\pi}{8}\left[\ln(u)\right]_{\square}^{0} \rightsquigarrow \frac{\pi}{8}\left[\ln(1+\rho^4)\right]_0^3 = \boxed{\frac{\pi}{8}\ln(1+3^4)}$$

(4) Exercise 6.2.27. Let D be a triangle in the (x,y) plane with vertices $(0,0)$, $(\frac{1}{2},\frac{1}{2})$, $(1,0)$. Evaluate: $\iint_D \cos\pi\left(\frac{x-y}{x+y}\right) dx\, dy$.

Solution: Let us make the change of variables:

$$\begin{cases} u = x+y \\ v = x-y. \end{cases}$$

this is a linear transformation, So we know how it behaves. Let's plot:



So the integral is, according to the change of variables formula:

$$\iint_D \cos\pi\left(\frac{x-y}{x+y}\right)dx\,dy = \iint_{D^*} \cos\pi\left(\frac{v}{u}\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du.$$

We need to compute the Jacobian. For that, let us compute $x(u,v)$ and $y(u,v)$.

$$\begin{cases} u = x+y \Rightarrow x = u-y \\ v = x-y \Rightarrow y = x-v \end{cases} \Rightarrow \begin{array}{l} x = u-(x-v) \Rightarrow \boxed{x=\frac{1}{2}(u+v)} \\ y = \frac{1}{2}(u+v)-v \Rightarrow \boxed{y=\frac{1}{2}(u-v)} \end{array}$$

. Hence, the Jacobia is.

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4}-\frac{1}{4} = -\frac{2}{4} = -\frac{1}{2}.$$

Take the absolute value $|J| = \frac{1}{2}$.

Now we can compute the integral:

$$\iint_{D^*} \cos\pi\left(\frac{v}{u}\right)\frac{1}{2}dv\,du = \frac{1}{2}\int_0^1\int_0^u \cos\left(\frac{\pi v}{u}\right)dv\,du = \frac{1}{2}\int_0^1 \left[\frac{u}{\pi}\sin\left(\frac{\pi v}{u}\right)\right]_0^u du$$

$$= \frac{1}{2}\int_0^1 \frac{u}{\pi}[\sin(\pi)-\sin(0)]\,du = \frac{1}{2}\int_0^1 \frac{u}{\pi}(0)\,du = \boxed{0}$$

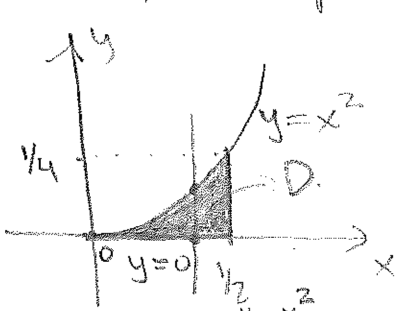
(5) Exercise 6.3.6. Find the center of mass of the region between: $y=0$ and $y=x^2$, where $0 \le x \le \frac{1}{2}$.

<u>Solution</u>: the center of mass $(\bar{x},\bar{y})$ is given by:

$$\bar{x} = \frac{\iint_D x\,\delta(x,y)\,dx\,dy}{\iint_D \delta(x,y)\,dx\,dy} \;;\; \bar{y} = \frac{\iint_D y\,\delta(x,y)\,dx\,dy}{\iint_D \delta(x,y)\,dx\,dy}.$$

Assuming constant density $\delta(x,y)=1 \;\forall x,y$

Let us first compute the area of the region:



the area is given by:

$$A = \iint_D dx\,dy = \int_0^{1/2}\int_0^{x^2} dy\,dx = \int_0^{1/2} x^2\,dx = \left[\frac{x^3}{3}\right]_0^{1/2}$$

$$\Rightarrow \boxed{A = \frac{1}{24}}$$

Now, $\bar{x} = 24\int_0^{1/2}\int_0^{x^2} x\,dy\,dx = 24\int_0^{1/2} x^3\,dx = 6[x^4]_0^{1/2} = 6\cdot\frac{1}{16} = \frac{3}{8} \Rightarrow \boxed{\bar{x} = \frac{3}{8}}$

$\bar{y} = 24\int_0^{1/2}\int_0^{x^2} y\,dy\,dx = 12\int_0^{1/2}[y^2]_0^{x^2}dx = 12\int_0^{1/2} x^4\,dx = \frac{12}{5}[x^5]_0^{1/2} = \frac{12}{5}\cdot\frac{1}{32} \Rightarrow \boxed{\bar{y} = \frac{3}{40}}$

So the center of mass of this region is $(\bar{x},\bar{y}) = \left(\frac{3}{8}, \frac{3}{40}\right)$

(6) Exercise 6.3.7. A sculptured gold plate D is defined by $0 \le x \le 2\pi$ and $0 \le y \le \pi$ (centimeters) and has mass density $\delta(x,y) = y^2 \sin^2(4x) + 2$ (g/cm^2). If gold sells for \$7 per gram, how much is the gold in the plate worth?

Solution: We want to compute the mass, in grams, of the plate:

$$\iint_D \delta(x,y)\,dy\,dx = \int_0^{2\pi}\int_0^{\pi} y^2 \sin^2(4x) + 2\,dy\,dx = \int_0^{2\pi} \left[\sin^2(4x)\frac{y^3}{3} + 2y\right]_0^{\pi} dx$$

$$= \int_0^{2\pi} \left[\sin^2(4x)\frac{\pi^3}{3} + 2\pi\right] dx = \left[\frac{\pi^3}{48}(8x - \sin(8x)) + 2\pi x\right]_0^{2\pi}$$

$$= \frac{\pi^3}{48}(16\pi - \cancel{\sin(16\pi)}) + 4\pi^2 = \frac{16}{48}\pi^4 + 4\pi^2 = \frac{\pi^4 + 12\pi^2}{3}$$

So the mass on the plate is $\boxed{m = \frac{\pi^4 + 12\pi^2}{3} \text{ grams}}$

to get the value, multiply by the rate \$7 per gram:

$$\text{Value} = 7\,\frac{\$}{\text{gram}} \times \frac{\pi^4 + 12\pi^2}{3}\,\text{grams} = \frac{7}{3}(\pi^4 + 12\pi^2)\,\$ \approx \boxed{503.6368\ \$}$$

(7) Exercise 6.3.11. Find the mass of the solid ball of radius 5 with density $\delta(x,y,z) = 2x^2 + 2y^2 + 2z^2 + 1$, centered at the origin.

Solution: the mass is given by (changing to spherical coordinates):

$$M = \int_0^{2\pi}\int_0^{\pi}\int_0^{5} (2\rho^2+1)(\rho^2 \sin\varphi)\,d\rho\,d\varphi\,d\theta = 2\pi \int_0^{\pi}\int_0^{5} 2\rho^4 \sin\varphi + \rho^2 \sin\varphi\,d\rho\,d\varphi$$

$$= 2\pi \int_0^{\pi} \left[\frac{2}{5}\rho^5 \sin\varphi + \frac{\rho^3}{3}\sin\varphi\right]_0^5 d\varphi = \int_0^{\pi} 2\cdot 5^4 \sin\varphi + \frac{5^3}{3}\sin\varphi\,d\varphi$$

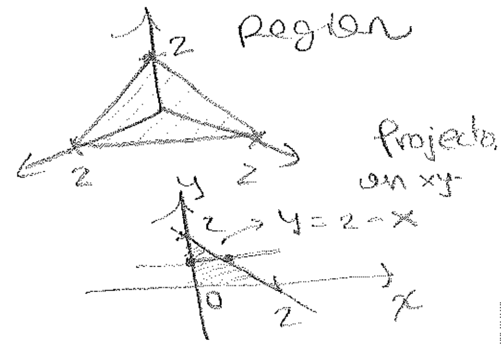
$$= 2\pi \int_0^{\pi} \sin\varphi\left(2\cdot 5^4 + \frac{5^3}{3}\right) d\varphi = 2\pi\left(2\cdot 5^4 + \frac{5^3}{3}\right)\int_0^{\pi} \sin\varphi\,d\varphi$$

$$= 2\pi\left(2\cdot 5^4 + \frac{5^3}{3}\right)\left[-\cos\varphi\right]_0^{\pi} = 4\pi\left(2\cdot 5^4 + \frac{5^3}{3}\right) = 4\pi\left(\frac{6\cdot 5^4 + 5^3}{3}\right)$$

$$= \boxed{\frac{4\pi}{3}(3875)}$$

(8) Exercise 6.3.13

Find the center of mass of the region bounded by $x+y+z=2$, $x=0$, $y=0$, $z=0$, assuming the density to be uniform.



Solution: First, let us compute the volume of the region

$$\int_0^2\int_0^{2-y}\int_0^{2-x-y} dz\,dx\,dy = \int_0^2\int_0^{2-y} 2-x-y\,dx\,dy$$

$$= \int_0^2 \left[2x-\frac{x^2}{2}-xy\right]_0^{2-y} dy$$

$$= \int_0^2 4-2y-\frac{(2-y)^2}{2}-(2-y)y\,dy = \int_0^2 4-2y-\frac{4-4y+y^2}{2}-2y+y^2\,dy$$

$$= \left[4y-y^2-2y+y^2-\frac{y^3}{6}-y^2+\frac{y^3}{3}\right]_0^2 = \left[-y^2+2y+\frac{y^3}{6}\right]_0^2 = \left(-4+4+\frac{8}{6}\right) = \boxed{\frac{8}{6}}$$

the center of mass is given by:

$$\bar{x} = \frac{6}{8}\int_0^2\int_0^{2-y}\int_0^{2-x-y} x\,dz\,dx\,dy = \frac{6}{8}\int_0^2\int_0^{2-y} x(2-x-y)\,dx\,dy = \frac{6}{8}\int_0^2\int_0^{2-y} 2x-x^2-xy\,dx\,dy$$

$$= \frac{6}{8}\int_0^2 \left[x^2-\frac{x^3}{3}-\frac{x^2y}{2}\right]_0^{2-y} dy = \frac{6}{8}\int_0^2 (2-y)^2-\frac{(2-y)^3}{3}-\frac{(2-y)^2y}{2}\,dy$$

$$= \frac{6}{8}\int_0^2 4-4y+y^2-\frac{(2-y)^3}{3}-\frac{4y-4y^2+y^3}{2}\,dy$$

$$= \frac{6}{8}\left[4y-2y^2+\frac{y^3}{3}+\frac{(2-y)^4}{12}-\frac{1}{2}\left(2y^2-\frac{4}{3}y^3+\frac{y^4}{4}\right)\right]_0^2$$

$$= \frac{6}{8}\left[4y-2y^2+\frac{y^3}{3}+\frac{(2-y)^4}{12}-y^2+\frac{2}{3}y^3-\frac{y^4}{8}\right]_0^2$$

$$= \frac{6}{8}\left[\left(\cancel{8}-\cancel{8}+\frac{8}{3}-4+\frac{16}{3}-2\right)-\frac{16}{12}\right] = \frac{6}{8}\left[\frac{24}{3}-6-\frac{16}{12}\right] = \frac{6}{8}\left[2-\frac{16}{12}\right] = \frac{6}{8}\left[\frac{24-16}{12}\right]$$

$$= \frac{6}{\cancel{8}}\cdot\frac{\cancel{8}}{12} = \boxed{\frac{1}{2}}$$

By symmetry, this point will also be the center of mass for y, z, i.e., $\bar{x}=\bar{y}=\bar{z}=1/2$, so the center of mass is

$$\boxed{(\bar{x},\bar{y},\bar{z}) = (1/2, 1/2, 1/2)}$$

(9) Exercise 6.3.16. Find the average value of e^{-z} over the ball $x^2+y^2+z^2 \le 1$.

Solution: By definition, the average of $f(x,y,z)=e^{-z}$ is

$$[f]_{av} = \frac{\iiint_W f(x,y,z)\,dx\,dy\,dz}{\iiint_W dx\,dy\,dz}$$

; But in this case we know the volume of the unit sphere is $\frac{4}{3}\pi$; Hence

$$[f]_{av} = \frac{3}{4\pi}\iiint_W f(x,y,z)\,dx\,dy\,dz. = \frac{3}{4\pi}\iiint_W e^{-z}\,dz\,dx\,dy$$

Using Cavalieri's principle, we need to integrate a cross section of the sphere as z varies from -1 to 1. The cross-section is given by

$A(z)$ = area of circle of radius $z = \pi \cdot r^2 = \pi(\sqrt{1-z^2})^2 = \pi(1-z^2)$.

So the average is

$$\frac{3}{4\pi}\int_{-1}^{1} e^{-z}\pi(1-z^2)\,dz = \frac{3}{4}\int_{-1}^{1} e^{-z} - z^2e^{-z}\,dz = \frac{3}{4}\left[-e^{-z}+z^2e^{-z}+2ze^{-z}+2e^{-z}\right]_{-1}^{1}$$

$$= \frac{3}{4}\left[z^2e^{-z}+2ze^{-z}+e^{-z}\right]_{-1}^{1} = \frac{3}{4}\left[(e^{-1}+2e^{-1}+e^{-1}) - (e^{1}-2e^{1}+e^{1})\right]$$

$$= \frac{3}{4}[4e^{-1}] = \boxed{3e^{-1}}$$

(10) Exercise 6.3.17. A solid with constant density is bounded above by the plane $z=a$ and below by the cone described in spherical coordinates by $\varphi = K$, where K is a constant $0<K<\pi/2$. Set up an integral for its moment of inertia about the z axis.

Solution: By definition, the moment of inertia about the z-axis is given by:

$I_z = \iiint_W (x^2+y^2)\,\delta\,dx\,dy\,dz$; in this cas δ is constant $\Rightarrow I_z = \delta\iiint_W (x^2+y^2)\,dx\,dy\,dz$.

Working in spherical coordinates we get $I_z = \delta\iiint_W (\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta)\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$.

Hence, $I_z = \delta\int_0^{K}\int_0^{2\pi}\int_0^{a\sec\varphi} \rho^4\sin^3\varphi\,d\rho\,d\theta\,d\varphi$.