M312 - Fall 2013 - <u>HW5</u> - Enrique Areyan ~~[illegible]~~ 99+1 ①

(1)

19. (a) Find a parametrization for the hyperboloid: $x^2+y^2-z^2=25$.

<u>Solution</u>: If you fix the value of z, then you get a circle centered at the origin of radius $\sqrt{25+z^2}$; i.e., $x^2+y^2=25+z^2$. We know how to parametrize the circle: $x=r\cos\theta$; $y=r\sin\theta$. $0\le\theta\le 2\pi$. So now regard z as varying

To get the parametrization of the hyperboloid that we want:

$$\boxed{\Phi(z,\theta)=\left(\sqrt{25+z^2}\cos\theta, \sqrt{25+z^2}\sin\theta, z\right)} \quad \boxed{\begin{array}{l}-\infty < z < \infty, \\ 0\le\theta\le 2\pi.\end{array}}$$

(b) Find an expression for a unit normal to this surface.

<u>Solution</u>: $\Phi_z = \left(\dfrac{z\cos\theta}{\sqrt{25+z^2}}, \dfrac{z\sin\theta}{\sqrt{25+z^2}}, 1\right)$

$\Phi_\theta = \left(-\sqrt{25+z^2}\sin\theta, \sqrt{25+z^2}\cos\theta, 0\right)$,

So the normal vector is given by:

$$\Phi_z\times\Phi_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{z\cos\theta}{\sqrt{25+z^2}} & \frac{z\sin\theta}{\sqrt{25+z^2}} & 1 \\ -\sqrt{25+z^2}\sin\theta & \sqrt{25+z^2}\cos\theta & 0 \end{vmatrix} = \hat{i}\left(-\sqrt{25+z^2}\cos\theta\right)-\hat{j}\left(\sqrt{25+z^2}\sin\theta\right)+\hat{k}\left(z\cos^2\theta+z\sin^2\theta\right)$$

$\Rightarrow \vec{n}=\langle -\sqrt{25+z^2}\cos\theta, -\sqrt{25+z^2}\sin\theta, z\rangle$

Next, normalize $\vec{n}$. $\Rightarrow \|\vec{n}\|=\sqrt{(25+z^2)\cos^2\theta+(25+z^2)\sin^2\theta+z^2}=\sqrt{25+2z^2}$

So, an expression for a unit normal to this surface is:

$$\boxed{\hat{n}=\langle -\sqrt{25+z^2}\cos\theta, -\sqrt{25+z^2}\sin\theta, z\rangle \frac{1}{\sqrt{25+2z^2}}}$$

(c) Find an equation for the plane tangent to the surface at $(x_0,y_0,0)$, where $x_0^2+y_0^2=25$, at the level curve $z=0$ is a vector parallel to $\vec{n}$, the normal vector to the plane.

<u>Solution</u>: the gradient of $f(x,y,z)=x^2+y^2-z^2=25$. But then $\nabla f(x,y,z)=\langle 2x,2y,-2z\rangle$, evaluate at $(x_0,y_0,0)$:

$\nabla f(x_0,y_0,0)=\langle 2x_0,2y_0,0\rangle$. Hence, the equation of the plane is

$\nabla f(x_0,y_0,0)\cdot\langle x-x_0, y-y_0, z-0\rangle=0$

$\langle 2x_0,2y_0,0\rangle\cdot\langle x-x_0,y-y_0,z\rangle=0$

$2x_0x-2x_0^2+2y_0y-2y_0^2=0 \iff x_0x+y_0y-(x_0^2+y_0^2)=0$

$\iff \boxed{x_0x+y_0y=25}$

d) SHow that the lines $(x_0, y_0, 0) + t(-y_0, x_0, 5)$ and $(x_0, y_0, 0) + t(y_0, -x_0, 5)$ lie in the surface and in the angent plane found in part (c).

olution: the lines lies in the surface since they satisfy the surface's eq:

$$l_1(t) = (x_0, y_0, 0) + t(-y_0, x_0, 5) = \langle x_0 - ty_0,\ y_0 + tx_0,\ 5t \rangle.$$

$$l_2(t) = (x_0, y_0, 0) + t(y_0, -x_0, 5) = \langle x_0 + ty_0,\ y_0 - tx_0,\ 5t \rangle. \quad \text{then,}$$

$$l_1(t):\ (x_0 - ty_0)^2 + (y_0 + tx_0)^2 - (5t)^2 = x_0^2 \cancel{- 2tx_0y_0} + t^2y_0^2 + y_0^2 \cancel{+ 2tx_0y_0} + t^2x_0^2 - 25t^2$$
$$= (x_0^2 + y_0^2) + t^2(x_0^2 + y_0^2) - 25t^2$$
$$= 25 + 25t^2 - 25t^2 = \boxed{25}$$

$$l_2(t):\ (x_0 + ty_0)^2 + (y_0 - tx_0)^2 - (5t)^2 = x_0^2 \cancel{+ 2tx_0y_0} + t^2y_0^2 + y_0^2 \cancel{- 2tx_0y_0} + t^2x_0^2 - 25t^2$$
$$= (x_0^2 + y_0^2) + t^2(x_0^2 + y_0^2) - 25t^2$$
$$= 25 + 25t^2 - 25t^2 = \boxed{25}$$

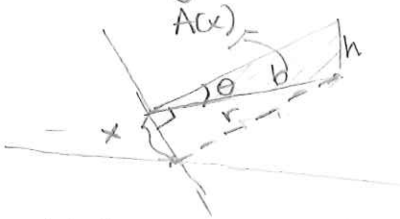
lso, the lines satisfy the equation of the tangent plane $x_0x + y_0y = 25$;

$$l_1(t):\ x_0(x_0 - ty_0) + y_0(y_0 + tx_0) = x_0^2 - tx_0y_0 + y_0^2 + tx_0y_0 = x_0 + y_0 = \boxed{25}$$

$$l_2(t):\ x_0(x_0 + ty_0) + y_0(y_0 - tx_0) = x_0^2 + tx_0y_0 + y_0^2 - tx_0y_0 = x_0 + y_0 = \boxed{25}$$

l_1 and l_2 lie in the tangent plane to the surface $x^2 + y^2 - z^2 = 25$.

5.1.7. By the Cavalieri's principle, Let us first compute the area $A(x)$ of cross-section of W, fixing x. then, $A(x)$ is the area of the triangle:



$$\tan\theta = \frac{h}{b} \Rightarrow \boxed{h = \tan\theta \cdot b}$$

$$r^2 = x^2 + b^2 \Rightarrow \boxed{b^2 = r^2 - x^2}$$

$$A(x) = \frac{h \cdot b}{2} = \frac{(\tan\theta \cdot b) \cdot b}{2} = \frac{\tan\theta\ b^2}{2} = \frac{\tan\theta\,(r^2 - x^2)}{2}.$$

o the volume is $W = \int_a^b A(x)dx$. But W is symmetric about the y-axis,

we can compute instead: $W = 2\int_0^r A(x)dx = \cancel{2}\int_0^r \frac{\tan\theta\,(r^2 - x^2)}{\cancel{2}}\,dx$

$$\tan\theta \int_0^r r^2 - x^2\,dx = \tan\theta\left[r^3 - \frac{x^3}{3}\right]_0^r = \tan\theta\left[r^3 - \frac{r^3}{3}\right] = \boxed{\frac{2}{3}\tan\theta\, r^3}$$

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(3) (a). (5.18) Show that the volume of the solid of revolution shown in figure 5.1.13 (a) is: $\pi \int_a^b [f(x)]^2 dx$.

Solution: By the Cavalieri's principle; the volume of the solid can be computed by fixing a cross-section and computing the corresponding area and then integrating, i.e., let W be the volume, then

$$W = \int_a^b A(x)dx. \quad (1)$$

But the area of a cross-section is just the area of the circle of radius $y=f(x)$. Therefore, $A(x)=\pi \cdot r^2 = \pi \cdot [f(x)]^2$ Replacing this into (1) we obtain the result:

$$W = \int_a^b \pi \cdot [f(x)]^2 dx = \pi \int_a^b [f(x)]^2 dx$$

(b) Show that the volume of the region obtained by rotating the region under the graph of the parabola $y=-x^2+2x+3$, $-1 \leq x \leq 3$, about the x axis is $512\pi/15$.

Solution: By the result obtained in part (a), we can compute this volume W as follow:

$$W = \int_{-1}^{3} \pi(-x^2+2x+3)^2 dx = \pi \int_{-1}^{3} (-x^2+2x)^2 + 6(-x^2+2x) + 9 \, dx =$$

$$\pi \int_{-1}^{3} x^4 - 4x^3 + 4x^2 - 6x^2 + 12x + 9 \, dx = \pi \int_{-1}^{3} x^4 - 4x^3 - 2x^2 + 12x + 9 \, dx$$

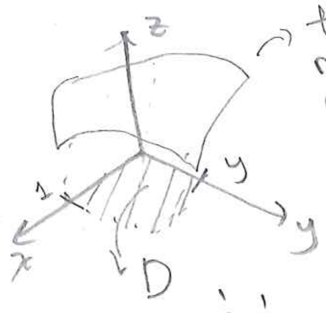
$$= \pi \left[\frac{x^5}{5} - x^4 - \frac{2}{3}x^3 + 6x^2 + 9x \right]_{-1}^{3} = \pi \left[\left(\frac{243}{5} - 81 - 18 + 54 + 27 \right) - \left(\frac{-1}{5} - 1 + \frac{2}{3} + 6 - 9 \right) \right]$$

$$= \pi \left[\left(\frac{243}{5} - 18 \right) - \left(\frac{2}{3} - \frac{1}{5} - 4 \right) \right] = \pi \left[\frac{243-90}{5} - \left(\frac{10-3-60}{15} \right) \right]$$

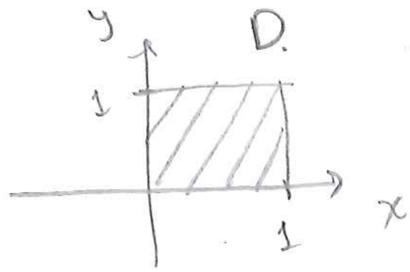
$$= \pi \left[\frac{153}{5} + \frac{53}{15} \right] = \pi \frac{459+53}{15} = \boxed{\frac{512}{15}\pi}$$

) 5.2.8. Compute the volume of the solid bounded by the xz plane, ... yz plane, the xy plane, the planes $x=1$ and $y=1$, and the surface $z=x^2+y^4$.

olution:



the surface may not look like this, this is only for reference.

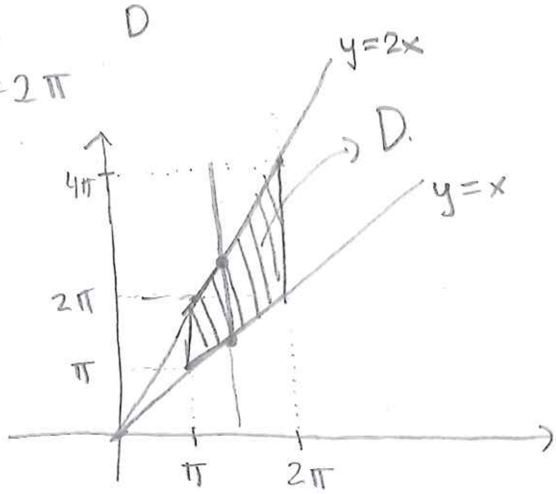


o the volume is: $\displaystyle\int_0^1\int_0^1 x^2+y^4\,dx\,dy = \int_0^1\left[\frac{x^3}{3}+xy^4\right]_0^1 dy$

$\displaystyle = \int_0^1 \frac{1}{3}+y^4\,dy = \left[\frac{1}{3}y+\frac{y^5}{5}\right]_0^1 = \frac{1}{3}+\frac{1}{5} = \frac{5+3}{15} = \boxed{\frac{8}{15}}$ //

) 5.3.12. Evaluate the following double integral: $\displaystyle\iint_D \cos y\,dx\,dy$, here the region D is bounded by $y=2x$, $y=x$, $x=\pi$, and $x=2\pi$

olution: Sketch of the region D:



$\displaystyle\iint_D \cos y\,dy\,dx = \int_{\pi}^{2\pi}\int_{x}^{2x}\cos y\,dy\,dx = \int_{\pi}^{2\pi}\left[\sin y\right]_x^{2x} dx$

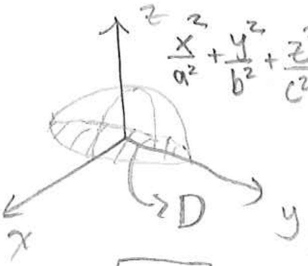
$\displaystyle \int_{\pi}^{2\pi} \sin(2x)-\sin(x)\,dx = \left[\frac{-\cos(2x)}{2}+\cos(x)\right]_{\pi}^{2\pi}$

$\displaystyle \left[\frac{-\cos(4\pi)}{2}+\cos(2\pi)\right]-\left[\frac{-\cos(2\pi)}{2}+\cos(\pi)\right]$

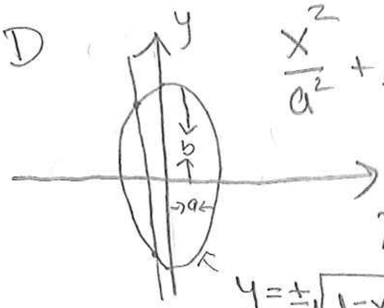
$\displaystyle \left[-\frac{1}{2}+1\right]-\left[-\frac{1}{2}-1\right] = -\cancel{\frac{1}{2}}+1+\cancel{\frac{1}{2}}+1 = \boxed{2}$ //

(6) 5.4.11. Compute the volume of an ellipsoid with semiaxes a, b, and c.

solution: Let us compute the volume of half the ellipsoid and multiply this result by 2.



$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$; the region of integration D in $\mathbb{R}^2$ is an ellipse →



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=$

$y=\pm\sqrt{1-\frac{x^2}{a^2}}\cdot b$

the integral of half the ellipsoid is:

$$\int_{-a}^{a}\int_{-\sqrt{1-\frac{x^2}{a^2}}\cdot b}^{\sqrt{1-\frac{x^2}{a^2}}\,b} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx = c\int_{-a}^{a}\int_{-\sqrt{1-\frac{x^2}{a^2}}\cdot b}^{\sqrt{1-\frac{x^2}{a^2}}\,b}\frac{1}{b}\sqrt{\left[b^2-\left(\frac{bx}{a}\right)^2\right]-y^2}\,dy\,dx =$$

(integral form 38, first page of book)

$$=\frac{c}{b}\int_{-a}^{a}\left[\frac{y}{2}\sqrt{b^2-\left(\frac{bx}{a}\right)^2-y^2}+\frac{b^2-\left(\frac{bx}{a}\right)^2}{2}\arcsin\left(\frac{y}{\sqrt{b^2-\left(\frac{bx}{a}\right)^2}}\right)\right]_{-\sqrt{1-\frac{x^2}{a^2}}\cdot b=-\sqrt{b^2-\left(\frac{bx}{a}\right)^2}}^{\sqrt{1-\frac{x^2}{a^2}}\cdot b=\sqrt{b^2-\left(\frac{bx}{a}\right)^2}}dx$$

$$=\frac{c}{b}\int_{-a}^{a}\left[\left(\frac{1}{2}\sqrt{b^2-\left(\frac{bx}{a}\right)^2}\,\overbrace{\sqrt{\cancel{b^2}-\cancel{\left(\frac{bx}{a}\right)^2}-\cancel{b^2}+\cancel{\left(\frac{bx}{a}\right)^2}}}^{0}+\frac{b^2-\left(\frac{bx}{a}\right)^2}{2}\arcsin\left(\frac{\cancel{\sqrt{b^2-\left(\frac{bx}{a}\right)^2}}}{\cancel{\sqrt{b^2-\left(\frac{bx}{a}\right)^2}}}\right)\right)-\right.$$

$$\left.\left(-\frac{1}{2}\sqrt{b^2-\left(\frac{bx}{a}\right)^2}\,\overbrace{\sqrt{\cancel{b^2}-\cancel{\left(\frac{bx}{a}\right)^2}-\cancel{b^2}+\cancel{\left(\frac{bx}{a}\right)^2}}}^{0}+\frac{b^2-\left(\frac{bx}{a}\right)^2}{2}\arcsin\left(\frac{-\cancel{\sqrt{b^2-\left(\frac{bx}{a}\right)^2}}}{\cancel{\sqrt{b^2-\left(\frac{bx}{a}\right)^2}}}\right)\right)\right]dx$$

$$=\frac{c}{b}\int_{-a}^{a}\frac{b^2-\left(\frac{bx}{a}\right)^2}{2}\left(\arcsin(1)-\arcsin(-1)\right)dx$$

$$=\frac{\cancel{b^2}c}{2\cancel{b}}\int_{-a}^{a}1-\left(\frac{x}{a}\right)^2\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)dx$$

$$=\frac{\pi bc}{2}\int_{-a}^{a}1-\frac{x^2}{a^2}\,dx=\frac{\pi bc}{2}\left[x-\frac{x^3}{3a^2}\right]_{-a}^{a}=\frac{\pi bc}{2}\left[\left(a-\frac{a^{\cancel{3}}}{3a^{\cancel{2}}}\right)-\left(-a+\frac{a^{\cancel{3}}}{3a^{\cancel{2}}}\right)\right]$$

$$=\frac{\pi bc}{2}\left[\left(a-\frac{a}{3}\right)-\left(-a+\frac{a}{3}\right)\right]=\frac{\pi bc}{2}\left[\frac{2}{3}a+\frac{2}{3}a\right]=\frac{\pi bc}{2}\left[\frac{4}{3}a\right]=\frac{4}{3}\left(\frac{abc\pi}{2}\right).$$

Finally, multiply by 2 to get the entire ellipsoid: $\boxed{\frac{4}{3}abc\pi}$

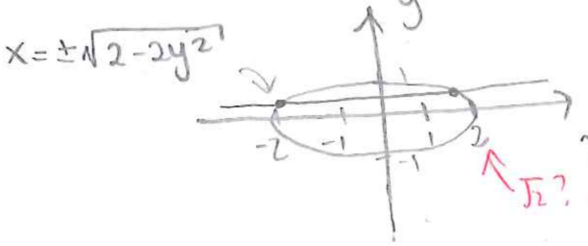
) 5.5.12. Find the volume of the solid bounded by

$$x^2+2y^2=2,\ z=0,\ \text{and}\ x+y+2z=2.$$

olution: z changes from 0 to $z=(2-x-y)/2$.

he projection of the surface $x^2+2y^2=2$ onto the xy plane yield an ellipse.

$$x^2+2y^2=2 \iff \frac{x^2}{2}+\frac{y^2}{1}=1$$



$\sqrt{2}$? Anyway no problem, does not affect your calculations

So the volume is given by:

$$\int_{-1}^{1}\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}}\int_{0}^{(2-x-y)/2} dz\,dx\,dy=\int_{-1}^{1}\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}}\frac{2-x-y}{2}\,dx\,dy=\frac{1}{2}\int_{-1}^{1}\left[2x-\frac{x^2}{2}-xy\right]_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}}dy$$

$$\frac{1}{2}\int_{-1}^{1}\left(2\sqrt{2-2y^2}-\cancel{\frac{2-2y^2}{2}}-y\sqrt{2-2y^2}\right)-\left(-2\sqrt{2-2y^2}-\cancel{\frac{2-2y^2}{2}}+y\sqrt{2-2y^2}\right)dy$$

$$\underbrace{\int_{-1}^{1}4\sqrt{2-2y^2}\,dy}_{A}-\underbrace{\int_{-1}^{1}2y\sqrt{2-2y^2}\,dy}_{B}.$$ Solve each separately:

$$A=4\sqrt{2}\int_{-1}^{1}\sqrt{1-y^2}\,dy=4\sqrt{2}\left[\frac{y}{2}\sqrt{1-y^2}+\frac{1}{2}\arcsin\left(\frac{y}{1}\right)\right]_{-1}^{1}=4\sqrt{2}\left[\left(\cancel{\tfrac{1}{2}\cdot 0}+\tfrac{1}{2}\arcsin(1)\right)-\left(\cancel{\tfrac{-1}{2}\cdot 0}+\tfrac{1}{2}\arcsin(-1)\right)\right]$$

$$4\sqrt{2}\left[\tfrac{1}{2}\arcsin(1)-\tfrac{1}{2}\arcsin(-1)\right]=4\sqrt{2}\left[\tfrac{1}{2}(\arcsin(1)-\arcsin(-1)\right]=2\sqrt{2}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=\boxed{2\pi\sqrt{2}}$$

$$B=2\sqrt{2}\int_{-1}^{1}y\sqrt{1-y^2}\,dy\ ;$$ substitute: $u=1-y^2 \Rightarrow du=-2y\,dy \Rightarrow y\,dy=\frac{du}{-2}$.

$$\to 2\sqrt{2}\int_0^0\frac{du}{-2}\sqrt{u}=-\sqrt{2}\int_0^0\sqrt{u}\,du=-\sqrt{2}\left[\frac{2}{3}u^{3/2}\right]_0^0 \rightsquigarrow -\frac{2}{3}\sqrt{2}\left[(1-y^2)^{3/2}\right]_{-1}^{1}$$

$$=-\frac{2}{3}\sqrt{2}\left[(1-1)^{3/2}-(1-1)^{3/2}\right]=0.$$

herefore, the volume is $A-B=2\pi\sqrt{2}-0=\boxed{2\pi\sqrt{2}}$

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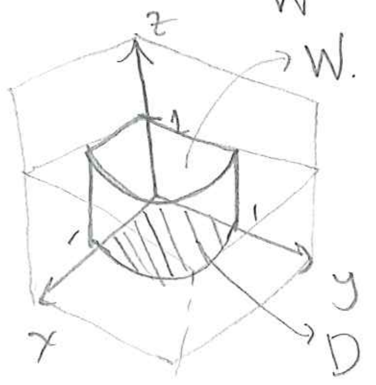
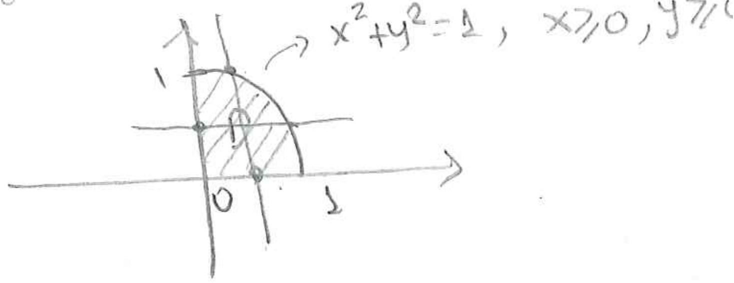
(8) 5.5.18. $\iiint_W z\,dx\,dy\,dz$; where W is the region bounded by

$x^2+y^2=1,\ x\geq 0,\ y\geq 0$
$x=0$
$y=0$
$z=0$
$z=1$



the projection of the cylinder onto $z=0$ is the region D, the quarter of the unit circle in the first quadrant.



$$\iiint_W z\,dx\,dy\,dz = \iiint_W z\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^1 z\,dz\,dy\,dx = \int_0^1\int_0^{\sqrt{1-x^2}} \left[\frac{z^2}{2}\right]_0^1 dy\,dx = \int_0^1\int_0^{\sqrt{1-x^2}} \frac{1}{2}\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 [y]_0^{\sqrt{1-x^2}}\,dx = \frac{1}{2}\int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4}\left[\sqrt{1-x^2}\,x + \arcsin(x)\right]_0^1$$

$$= \frac{1}{4}\left[\left(\cancel{\sqrt{1-1}\,1} + \arcsin(1)\right) - \left(\cancel{\sqrt{1-0}\,0} + \arcsin(0)\right)\right]$$

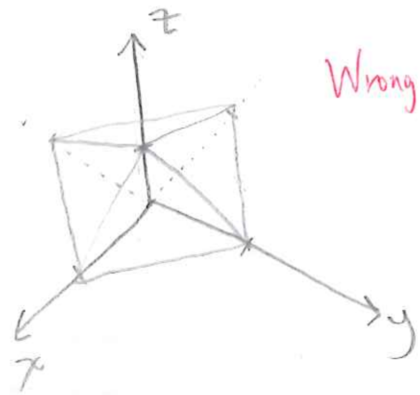
$$= \frac{1}{4}\left[\arcsin(1) - \arcsin(0)\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{2}\right]$$

$$= \boxed{\frac{\pi}{8}}$$

xtra Credit: Let P denote the pyramid with vertices at $(\pm 1, \pm 1, 0)$ and
$(0,0,1)$ and for $a > 0$ let B be the ball given by $x^2 + y^2 + (z-a)^2 \leq a^2$.
By $B \setminus P$ denote the set of those points of the ball B that lie outside of
he pyramid P. Find all the values of a (they form an interval) for
hich $B \setminus P$ consists of four disjoint parts. Find the volume of the common
rt $P \cap B$ for those a.

olution:



Wrong. First note that if $0 \leq a \leq \frac{1}{2}$ then $B \setminus P = \Phi$, i.e., the ball lies entirely inside the pyramid.

If $\frac{1}{2} < a < 1$, then there is an intersection between P and B.

If $a > 1$ then the ball is outside the pyramid $B \setminus P = B$.

So the set of points for a is
$a \in (\frac{1}{2}, 1)$.

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