M312: Fall 2013 - Enrique Areyan - HW 4 99 + ~~10~~ 12

(1) For which $a \in \mathbb{R}$ the vector field $F(x,y) = (e^{x+y} + ay, e^{x+y} + x)$ is a gradient field on $\mathbb{R}^2$? For those a find a scalar function f with $F = \nabla$

Solution: By theorem proved on class, F is a gradient field if:

$$\frac{\partial}{\partial y} F_1 = \frac{\partial}{\partial x} F_2 \iff \frac{\partial}{\partial y}(e^{x+y} + ay) = \frac{\partial}{\partial x}(e^{x+y} + x)$$

$$\iff e^{x+y} + a = e^{x+y} + 1 \iff \boxed{a = 1}$$

So, the only value of a for which $F(x,y)$ is a gradient field is $a=$

Now, find the potential function $f: \mathbb{R}^2 \to \mathbb{R}$ of the gradient field:

$$F(x,y) = (e^{x+y} + y, e^{x+y} + x)$$

f satisfies the following:

$$\frac{\partial f}{\partial x} = e^{x+y} + y \Rightarrow f(x,y) = \int \frac{\partial f}{\partial x} \partial x = \int (e^{x+y} + y) \partial x = e^{x+y} + xy + g(y),$$

where $g(y)$ is a pure function of

But then,

$$\frac{\partial f}{\partial y} = e^{x+y} + y = \frac{\partial}{\partial y}\left(e^{x+y} + xy + g(y)\right) = e^{x+y} + y + g'(y).$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = c, \text{ for } c \text{ a constant.}$$

Hence, the function f is: $\boxed{f(x,y) = e^{x+y} + xy + c}$

We can check that indeed: $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle e^{x+y} + y, e^{x+y} + x \rangle = F(x,$

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(2) Exercise 7.2.11 Evaluate $\int_C F \cdot ds$, where $F(x,y) = x\hat{\imath} + y\hat{\jmath}$, and th

curve is $c(t) = \langle \cos^3 t, \sin^3 t \rangle$, $0 \leq t \leq 2\pi$.

Solution: Note that $F(x,y)$ is a gradient field since: $F(x,y) = \langle x, y \rangle$ is such that: $\frac{\partial}{\partial y} F_1 = \frac{\partial x}{\partial y} = 1 = \frac{\partial y}{\partial x} = \frac{\partial F_2}{\partial x}$. So we can solve for f s.t. $F = \nabla f$

should be 0

f satisfies the following:

$$\frac{\partial f}{\partial x} = x \Rightarrow f(x,y) = \int \frac{\partial f}{\partial x} \partial x = \int x \partial x = \frac{x^2}{2} + g(y),$$ where $g(y)$ is a pure function of y.

3ut then: $\frac{\partial f}{\partial y} = y = \frac{\partial}{\partial y}\left(\frac{x^2}{2}+g(y)\right) = g'(y)$

$\Rightarrow g'(y)=y \quad \Rightarrow \quad g(y)=\int g'(y)\,dy = \int y\,dy = \frac{y^2}{2}+c$, for c a constant.

ence, the function f is : $\boxed{f(x,y)=\frac{x^2}{2}+\frac{y^2}{2}+c}$

ote that indeed $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (x,y) = F(x,y)$.

herefore, by theorem 3 of line integrals of gradient vector fields, we have

$\int_c F\cdot ds = \int_c \nabla f\cdot ds = f(c(2\pi)) - f(c(0)) = f(1,0)-f(1,0) = \boxed{0}$

) Exercise 7.2.14. Let $F = \langle z^3+2xy, x^2, 3xz^2\rangle$. Show that the integral of F around the circumference of the unit square with vertices $(\pm1,\pm1)$ is zero.

ution: Note that curl F = 0, so F is a gradient field.

$$\text{rl } F = \nabla\times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^3+2xy & x^2 & 3xz^2 \end{vmatrix} = \hat{\imath}\left(\frac{\partial}{\partial y}(3xz^2)-\frac{\partial}{\partial z}(x^2)\right) - \hat{\jmath}\left(\frac{\partial}{\partial x}(3xz^2)-\frac{\partial}{\partial z}(z^3+2xy)\right) + \hat{k}\left(\frac{\partial}{\partial x}(x^2)-\frac{\partial}{\partial y}(z^3+2xy)\right)$$

$$= \hat{\imath}(0-0) - \hat{\jmath}(3z^2-3z^2) + \hat{k}(2x-2x) = 0\hat{\imath}-0\hat{\jmath}-0\hat{k} = \boxed{\vec{0}}$$

ence, we can find the potential f of F.

$\frac{\partial f}{\partial x} = z^3+2xy \Rightarrow f(x,y,z) = \int (z^3+2xy)\,\partial x = xz^3+x^2y+g(y,z)$.

$\frac{\partial f}{\partial y} = x^2 = \frac{\partial}{\partial y}(xz^3+x^2y+g(y,z)) = x^2+\frac{\partial}{\partial y}g \Rightarrow \frac{\partial}{\partial y}g = 0 \Rightarrow g(y,z)=h(z)$

o far we have $f(x,y,z) = xz^3+x^2y+h(z)$. Using the final condition:

$\frac{\partial f}{\partial z} = 3xz^2 = \frac{\partial}{\partial z}(xz^3+x^2y+h(z)) = 3xz^2+h'(z) \Rightarrow h'(z)=0 \Rightarrow h(z)=c$, c a constant

us, f is given by $\boxed{f(x,y,z)=xz^3+x^2y+c}$

erefore, by theorem 3 of line integrals of gradient vector fields, we have:

$\int_c F\,ds = \int_c \nabla f\,ds = f(c(b))-f(c(a))$; where c is a parametrization of the circumference of the unit square with vertices $(\pm1,\pm1)$. Any such parametrization will have the same start and end points since the square is a closed path.

erefore, $c(b)=c(a)$ and $\int_c F\,ds = \int_c \nabla f\,ds = f(c(b))-f(c(a)) = f(c(b))-f(c(b)) = \boxed{0}$

(4) For a continuous vector field F on a path c show that: ②

$$\left| \int_c F \cdot ds \right| \leq \int_c \| F \circ c \| ds.$$

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Pf: the proof uses two facts:

Ⓘ Cauchy-Schwarz Inequality: $|u \cdot v| \leq \|u\| \|v\|$, for vectors u, v

Ⓘⓘ $\int_a^b |f(x)| dx \geq \left| \int_a^b f(x) dx \right|$

$$|F(c(t)) \cdot c'(t)| \leq \|F(c(t))\| \cdot \|c'(t)\|$$, by Ⓘ for any t

$$\int_a^b |F(c(t)) \cdot c'(t)| dt \leq \int_a^b \|F(c(t))\| \cdot \|c'(t)\| dt$$, integrating both sides

$$\left| \int_a^b F(c(t)) \cdot c'(t) dt \right| \leq \int_a^b |F(c(t)) \cdot c'(t)| dt \leq \int_a^b \|F(c(t))\| \cdot \|c'(t)\| dt$$, by Ⓘⓘ

$$\Rightarrow \left| \int_a^b F(c(t)) \cdot c'(t) dt \right| \leq \int_a^b \|F(c(t))\| \|c'(t)\| dt$$

But these are precisely the definitions of line and path integral respective

$$\int_a^b F(c(t)) \cdot c'(t) dt = \int_c F \cdot ds \ ; \quad \int_a^b \|F(c(t))\| \|c'(t)\| dt = \int_c \|F \circ c\| ds$$

therefore, $\left| \int_c F \cdot ds \right| \leq \int_c \|F \circ c\| ds$ //

(6) Exercise 7.2.7. Suppose the path c has lenght ℓ, and $\|F\| \leq M$. Prove that

$$\left| \int_c F \cdot ds \right| \leq M\ell.$$

Pf: the proof is very similar to that of (4), and I will refer to it:

$$|F(c(t)) \cdot c'(t)| \leq \|F(c(t))\| \cdot \|c'(t)\| \leq M \cdot \|c'(t)\|$$, by Ⓘ and $\|F\| \leq M$

$$\int_a^b |F(c(t)) \cdot c'(t)| dt \leq M \int_a^b \|c'(t)\| dt = M\ell$$, integrating and $L = \int_a^b \|c'(t)\|$

$$\Rightarrow \int_a^b |F(c(t)) \cdot c'(t)| \leq M\ell$$; but by Ⓘⓘ we get $\left| \int_a^b F(c(t)) \cdot c'(t) dt \right| \leq M\ell$.

Finally, by definition: $\boxed{\left| \int_c F \cdot ds \right| \leq M\ell}$ //

) Exercise 7.2.17. Evaluate the integral

$$\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz$$

where c is an oriented simple curve connecting $(1,1,1)$ to $(1,2,4)$.

olution: Probably the easiest parametrization of c is the line from $(1,1,1)$ to $(1,2,4)$, given by $c(t)=(1-t)(1,1,1)+t(1,2,4)=\langle 1, 1+t, 1+3t\rangle$; $0\le t\le 1$

hen, $dx=0$; $dy=1$; $dz=3$.

$$\int 2xyz\,dx+x^2z\,dy+x^2y\,dz = \int_0^1 \left[\cancel{2(1)(1+t)(1+3t)(0)}+(1)^2(1+3t)(1)+(1)^2(1+t)(3)\right]dt$$

$$=\int_0^1 (1+3t+3+3t)\,dt=\int_0^1 4+6t\,dt=\left[4t+3t^2\right]_0^1=4+3=\boxed{7}$$

) Exercise 7.2.18. Suppose $\nabla f(x,y,z)=\langle 2xyze^{x^2}, ze^{x^2}, ye^{x^2}\rangle$.

If $f(0,0,0)=5$, find $f(1,1,2)$.

lution: Let us find the potential f.

$$\frac{\partial f}{\partial x}=2xyze^{x^2} \Rightarrow f(x,y,z)=\int 2xyze^{x^2}\partial x = e^{x^2}yz+g(y,z)$$ (By a simple change of variables $u=e^{x^2}$)

$$\frac{\partial f}{\partial y}=ze^{x^2}=\frac{\partial}{\partial y}\left(e^{x^2}yz+g(y,z)\right)=e^{x^2}z+\frac{\partial}{\partial y}g \Rightarrow \frac{\partial g}{\partial y}=0 \Rightarrow g(y,z)=h(z).$$

o far we have $f(x,y,z)=e^{x^2}yz+h(z)$. Using the last condition:

$$\frac{\partial f}{\partial z}=ye^{x^2}=\frac{\partial}{\partial z}\left(e^{x^2}yz+h(z)\right)=e^{x^2}y+h'(z) \Rightarrow h'(z)=0 \Rightarrow h(z)=c,$$ for c a consta.

erefore, $$\boxed{f(x,y,z)=e^{x^2}yz+c}$$

eplacing:

$$f(0,0,0)=5=e^{0^2}\cdot 0\cdot 0+c=c \Rightarrow \boxed{c=5}$$

So our particular function is

$$\boxed{f(x,y,z)=e^{x^2}yz+5}$$

nally, $f(1,1,2)=e^{1^2}\cdot 1\cdot 2+5=\boxed{5+2e}$

(9) Exercise 7.3.3. Find an equation for the plane tangent to the given surface at the specified point.

$$x = u^2 \; ; \; y = u\sin(e^v) \; ; \; z = \frac{1}{3}u\cos(e^v) \text{ , at } (13,-2,1).$$

Solution: $\Phi(u,v) = \langle u^2, u\sin(e^v), \frac{1}{3}u\cos(e^v)\rangle$. First, find the normal vector to the desired plane: $\vec{n} = T_u \times T_v$, where

$$T_u = \frac{\partial \Phi}{\partial u} = \left\langle 2u, \sin(e^v), \tfrac{1}{3}\cos(e^v)\right\rangle \; ; \; T_v = \left\langle 0, ue^v\cos(e^v), -\tfrac{1}{3}ue^v\sin(e^v)\right\rangle$$

$$T_u \times T_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & \sin(e^v) & \frac{1}{3}\cos(e^v) \\ 0 & ue^v\cos(e^v) & -\frac{1}{3}ue^v\sin(e^v) \end{vmatrix} = \hat{i}\left(-\tfrac{1}{3}ue^v\sin^2(e^v) - \tfrac{1}{3}ue^v\cos^2(e^v)\right) - \hat{j}\left(-\tfrac{2}{3}u^2e^v\sin(e^v)\right) + \hat{k}\left(2u^2e^v\cos(e^v)\right)$$

$$= \hat{i}\left(-\tfrac{1}{3}ue^v\right) + \hat{j}\left(\tfrac{2}{3}u^2e^v\sin(e^v)\right) + \hat{k}\left(2u^2e^v\cos(e^v)\right)$$

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Hence, $\vec{n} = -\frac{1}{3}ue^v\langle 1, -2u\sin(e^v), -6u\cos(e^v)\rangle$, but any multiple of this vector is also a normal vector, so use $\vec{n} = \langle 1, -2u\sin(e^v), -6u\cos(e^v)\rangle$. Hence, we need to find (u_0, v_0) such that $\Phi(u_0,v_0) = (13,-2,1)$.

$$\Phi(u_0,v_0) = \langle u_0^2, u_0\sin(e^{v_0}), \tfrac{1}{3}u_0\cos(e^{v_0})\rangle = \langle 13,-2,1\rangle \Rightarrow$$

$u_0^2 = 13$; $u_0\sin(e^{v_0}) = -2$; $\frac{1}{3}u_0\cos(e^{v_0}) = 1$. this is complicated to solve, but note that it is very close to what we want for $\vec{n}$:

$$\left.\begin{array}{l} u_0\sin(e^{v_0}) = -2 \Rightarrow -2u_0\sin(e^{v_0}) = 4 \\ \frac{1}{3}u_0\cos(e^{v_0}) = 1 \Rightarrow -6u_0\cos(e^{v_0}) = -18 \end{array}\right\} \Rightarrow \vec{n} = \langle 1,4,-18\rangle.$$

So the equation of the plane is:

$$\vec{n}\cdot\langle x-x_0, y-y_0, z-z_0\rangle = 0 \iff$$

$$\langle 1,4,-18\rangle\cdot\langle x-13, y+2, z-1\rangle = 0 \iff$$

$$x-13+4y+8-18z+18 = 0 \iff$$

$$\boxed{x+4y-18z = -13}$$

Note that this equation does contain the point $(13,-2,1)$:

$$13+4(-2)-18(1) = 13-8-18 = -13.$$

0) Exercise 7.3.4. At what points are the following surfaces regular?

1. $x=2u$, $y=u^2+v$, $z=v^2$ $\Rightarrow \Phi(u,v)=\langle 2u, u^2+v, v^2\rangle$.

$T_u=\langle 2, 2u, 0\rangle$; $T_v=\langle 0,1,2v\rangle$.

$$T_u\times T_v=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 2u & 0\\ 0 & 1 & 2v\end{vmatrix}=\hat{i}(4uv)-\hat{j}(4v)+\hat{k}(2)=\langle 4uv,-4v,2\rangle\neq\langle 0,0,0\rangle$$

since $2\neq 0$,

Therefore, this surface is regular. (regular at all points).

2. $x=u^2-v^2$, $y=u+v$, $z=u^2+4v$ $\Rightarrow \Phi(u,v)=\langle u^2-v^2, u+v, u^2+4v\rangle$

$T_u=\langle 2u,1,2u\rangle$; $T_v=\langle -2v,1,4\rangle$.

$$T_u\times T_v=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2u & 1 & 2u\\ -2v & 1 & 4\end{vmatrix}=\hat{i}(4-2u)-\hat{j}(8u+4uv)+\hat{k}(2u+2v)=\langle 4-2u,-8u-4uv,2u+2v\rangle$$

The surface will not be regular when $\langle 4-2u,-8u-4uv,2u+2v\rangle=\langle 0,0,0\rangle \Rightarrow$

$4-2u=0 \Rightarrow \boxed{u=2}$
$8u+4uv=0 \Rightarrow -16+16=0$
$2u+2v=0 \Rightarrow 4+2v=0 \Rightarrow \boxed{v=-2}$

$\Rightarrow$ the surface is not regular at $(u_0,v_0)=(2,-2)$
$\Phi(2,-2)=\langle 0,0,-4\rangle$.
It is regular at all other points $\mathbb{R}^3\setminus\{(0,0,-4)\}$.

) Exercise 7.3.9. Find an expression for a unit vector normal to the surface
$x=\cos v\sin u$; $y=\sin v\sin u$; $z=\cos u$
the image of a point (u,v) for u in $[0,\pi]$ and v in $[0,2\pi]$. Identify this surface.

Solution: $\Phi(u,v)=\langle\cos v\sin u, \sin v\sin u, \cos u\rangle$.

$T_u=\langle\cos v\cos u, \sin v\cos u, -\sin u\rangle$; $T_v=\langle -\sin v\sin u, \cos v\sin u, 0\rangle$

$$T_u\times T_v=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \cos v\cos u & \sin v\cos u & -\sin u\\ -\sin v\sin u & \cos v\sin u & 0\end{vmatrix}=\hat{i}(\cos v\sin^2 u)-\hat{j}(-\sin v\sin^2 u)+\hat{k}(\cos^2 v\cos u\sin u+\sin^2 v\sin u\cos u)$$
$$=\langle\cos v\sin^2 u, \sin v\sin^2 u, \cos u\sin u\rangle$$
$$=\sin u\langle\cos v\sin u, \sin v\sin u, \cos u\rangle=\vec{n}.$$

Normalize $\vec{n}$: $\|\vec{n}\|=\sqrt{\sin^2 u(\cos^2 v\sin^2 u+\sin^2 v\sin^2 u+\cos^2 u)}$
$$=\sin u\sqrt{\sin^2 u(\cos^2 v+\sin^2 v)+\cos^2 u}=\sin u\sqrt{\sin^2 u+\cos^2 u}=\sin u.$$

Therefore, an expression for a unit normal vector is

$\hat{n}=\dfrac{\vec{n}}{\|\vec{n}\|}=\boxed{\langle\cos v\sin u, \sin v\sin u, \cos u\rangle}$; We saw in class that this is parametrization of the unit sphere centered at the origin

(12) Exercise 7.3.14. Find the equation of the plane tangent to the surface $x=u^2$, $y=v^2$, $z=u^2+v^2$, at the point $u=1, v=1$

Solution: Let $\Phi(u,v)=\langle u^2, v^2, u^2+v^2\rangle$. then

$T_u=\langle 2u, 0, 2u\rangle$; $T_v=\langle 0, 2v, 2v\rangle$.

?

$$T_u \times T_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & 0 & 2u \\ 0 & 2v & 2v \end{vmatrix} = \hat{i}(-4uv) - \hat{j}(4uv) + \hat{k}(0) = \langle -4uv, -4uv, 0\rangle$$
$$= -4uv\langle 1,1,0\rangle.$$

So a normal at $(u,v)=(1,1)$ is $\vec{n}(1,1)=-4(1)(1)\langle 1,1,0\rangle = \langle -4,-4,0\rangle$ which we can scale to obtain $\boxed{\vec{n}=\langle 1,1,0\rangle}$

the equation of the tangent plane is given by

$$\vec{n}\cdot\langle x-x_0, y-y_0, z-z_0\rangle = 0 \iff$$
$$\langle 1,1,0\rangle\cdot\langle x-1, y-1, z-2\rangle = 0 \iff$$
$$x-1+y-1=0 \iff \boxed{x+y=2}$$

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Note that this plane indeed contains: $\Phi(1,1)=\langle 1,1,2\rangle$.

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(5) Extra Credit. For $(x,y)\neq(0,0)$ define the vector field
$$F(x,y)=\frac{1}{e^y\sqrt{x^2+y^2}}(\cos x, \sin x).$$

For $R>0$, let $c(t)=R(\cos t, \sin t)$, $0\leq t\leq \pi/2$. Use problem 4 to prove
$$\lim_{R\to\infty}\int_c F\cdot ds = 0.$$

Pf: If we can show that: $\lim_{R\to\infty}\int_c \|F\circ c\| ds = 0$, then by problem 4:

$$\lim_{R\to\infty}\left|\int_c F\cdot ds\right| \leq \lim_{R\to\infty}\int_c \|F\circ c\| ds = 0 \Rightarrow \lim_{R\to\infty}\int_c F\cdot ds = 0.$$

o our goal is to show:

$$\lim_{R\to\infty} \int_C \|F\circ c\|\, ds \stackrel{?}{=} 0.$$

y definition:

$$F\circ c = F(c(t)) = F(R\cos t, R\sin t) = \frac{1}{e^{R\sin t}\cdot R}\langle \cos(R\cos t), \sin(R\cos t)\rangle$$

$$\|F\circ c\| = \sqrt{\left(\frac{1}{e^{R\sin t}\cdot R}\right)^2\left(\cos^2(R\cos t)+\sin^2(R\cos t)\right)} = \frac{1}{e^{R\sin t}\cdot R}.$$ Moreover,

$$\int_C \|F\circ c\|\, ds = \int_0^{\pi/2} \|F(c(t))\|\,\|c'(t)\|\, dt,$$ where $c'(t) = \langle -R\sin t, R\cos t\rangle$, $\|c'(t)\| = R$.

$$\Rightarrow \int_C \|F\circ c\|\, ds = \int_0^{\pi/2} \frac{1}{e^{R\sin t}\cdot \cancel{R}}\cancel{R}\, dt = \int_0^{\pi/2} \frac{1}{e^{R\sin t}}\, dt$$

herefore,

$$\lim_{R\to\infty} \int_C \|F\circ c\|\, ds = \lim_{R\to\infty} \int_0^{\pi/2} \frac{1}{e^{R\sin t}}\, dt,$$ since this function converges

What's meant by a function converges?

$$\lim_{R\to\infty} \int_0^{\pi/2} \frac{1}{e^{R\sin t}}\, dt = \int_0^{\pi/2} \lim_{R\to\infty} \frac{1}{e^{R\sin t}}\, dt = \int_0^{\pi/2} 0\, dt = 0,$$

hich shows the result.

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A possible approach:

Let $\varepsilon > 0$. ($\varepsilon < \frac{\pi}{2}$).

Then $$\int_0^{\frac{\pi}{2}} \frac{1}{e^{R\sin t}}\, dt = \int_0^{\varepsilon} \frac{1}{e^{R\sin t}}\, dt + \int_{\varepsilon}^{\frac{\pi}{2}} \frac{1}{e^{R\sin t}}\, dt$$

$$\leq \int_0^{\varepsilon} 1\, dt + \int_{\varepsilon}^{\frac{\pi}{2}} \frac{1}{e^{R\sin\varepsilon}}\, dt$$

$$\leq \varepsilon + \left(\frac{\pi}{2} - \varepsilon\right)\frac{1}{e^{R\sin\varepsilon}}$$

$$\to \varepsilon \quad \text{as } R\to\infty.$$

This needs some careful analysis ~~here~~ actually, but I omit it

$\therefore \left|\int_0^{\frac{\pi}{2}} \frac{1}{e^{R\sin t}}\, dt\right| \leq 2\varepsilon$ as R is big

$\therefore$ the limit is zero.