M.312 - Fall 2013 - HW2 - Enrique Areyan (100) ①

4.3.15. Show that the given curve $c(t)$ is a flow line of the given vector field $F(x,y,z)$.

$c(t) = (e^{2t}, \log|t|, 1/t),\ t \neq 0;\quad F(x,y,z) = (2x, z, -z^2)$.

Solution: By definition, If F is a vector field, a flow line for F is a path $c(t)$ such that $c'(t) = F(c(t))$.

In this case, $c'(t) = ((e^{2t})', (\log|t|)', (1/t)') = (2e^{2t}, \frac{1}{t}, -\frac{1}{t^2})$.

And: $F(c(t)) = F(e^{2t}, \log|t|, 1/t) = (2e^{2t}, \frac{1}{t}, -\frac{1}{t^2})$, which shows that $c(t)$ is a flow line of F.

$$c'(t) = (2e^{2t}, \tfrac{1}{t}, -\tfrac{1}{t^2}) = F(e^{2t}, \log|t|, 1/t) = F(c(t)).$$

4.3.19. Let $F(x,y,z) = (x^2, yx^2, z+zx)$ and $c(t) = (\frac{1}{1-t}, 0, \frac{e^t}{1-t})$.

SHOW $c(t)$ is a flow line for F.

Solution: We proceed as before:

$$c'(t) = \left(\frac{1}{(1-t)^2}, 0, \frac{e^t}{1-t} + \frac{e^t}{(1-t)^2}\right) = \left(\frac{1}{(1-t)^2}, 0\cdot\left(\frac{1}{1-t}\right), \frac{e^t}{1-t} + \frac{e^t}{1-t}\left(\frac{1}{1-t}\right)\right)$$

$$= F\left(\frac{1}{1-t}, 0, \frac{e^t}{1-t}\right) = F(c(t)),$$

which shows that $c(t)$ is a flow line of F.

4.3.21 (a). Let $F(x,y,z) = (yz, xz, xy)$. Find a function $f: \mathbb{R}^3 \to \mathbb{R}$ such that $F = \nabla f$.

Solution: Let f be a function $f: \mathbb{R}^3 \to \mathbb{R}$ such that $F = \nabla f$. then

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (yz, xz, xy) \iff \frac{\partial f}{\partial x} = yz;\ \frac{\partial f}{\partial y} = xz$$

and $\frac{\partial f}{\partial z} = xy$. We can integrate these functions and substitute appropiately to obtain a family of functions for f as follow:

$\frac{\partial f}{\partial x} = yz \Rightarrow f = \int \frac{\partial f}{\partial x} \partial x = \int yz \, \partial x = xyz + g(y,z)$

We use the conditions on f to simplify this expression

$\frac{\partial f}{\partial y} = xz = \frac{\partial}{\partial y}(xyz + g(y,z)) = xz + \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial y} = xz - xz = 0.$

Thus, $g(y,z) = \int \frac{\partial g}{\partial y} \cdot \partial y = \int 0 \, \partial y = C_1 + h(z)$; where C_1 is a constant.

So far we have: $f(x,y,z) = xyz + h(z) + C_1$.

Finally,

$\frac{\partial f}{\partial z} = xy = \frac{\partial}{\partial z}(xyz + h(z) + C_1) = xy + h'(z) \Rightarrow h'(z) = 0$

Thus, $h(z) = \int h'(z) \, dz = \int 0 \, dz = C_2$, C_2 a constant.

So the function f is: $\boxed{f(x,y,z) = xyz + C_3}$, C_3 a constant ($C_1 + C_2 = C_3$)

3.21 (b) Let $\mathbf{F}(x,y,z) = (x,y,z)$. Find a function $f: \mathbb{R}^3 \to \mathbb{R}$ s.t. $\mathbf{F} = \nabla f$.

Solution: Again, let f be such a function. then:

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (x,y,z). \Leftrightarrow \frac{\partial f}{\partial x} = x; \frac{\partial f}{\partial y} = y; \frac{\partial f}{\partial z} = z.$ then:

Proceed as before:

$\frac{\partial f}{\partial x} = x \Rightarrow f = \int \frac{\partial f}{\partial x} \partial x = \int x \, \partial x = \frac{x^2}{2} + g(y,z).$

$\frac{\partial f}{\partial y} = y = \frac{\partial}{\partial y}\left(\frac{x^2}{2} + g(y,z)\right) = \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial y} = y \Rightarrow g = \int \frac{\partial g}{\partial y} \partial y = \int y \, \partial y = \frac{y^2}{2} + h(z).$

$\Rightarrow g(y,z) = \frac{y^2}{2} + h(z)$. So far we have: $f(x,y,z) = \frac{x^2}{2} + \frac{y^2}{2} + h(z).$

Finally, $\frac{\partial f}{\partial z} = z = \frac{\partial}{\partial z}\left(\frac{x^2}{2} + \frac{y^2}{2} + h(z)\right) = h'(z) \Rightarrow h'(z) = z$, and so

$h(z) = \int h' dz = \int z \, dz = \frac{z^2}{2} + C$, C a constant.

Therefore, our function f is: $\boxed{f(x,y,z) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + C}$

4.3.24. Let $\mathbf{c}(t)$ be a flow line of a gradient field $\mathbf{F} = -\nabla V$. Prove that $V(\mathbf{c}(t))$ is a decreasing function of t.

Proof: We want to show that $\frac{d}{dt} V(\mathbf{c}(t)) \overset{?}{\leq} 0$. If that is the case then we can conclude that $V(\mathbf{c}(t))$ is decreasing. So, let us compute

$$\frac{d}{dt} V(\mathbf{c}(t)) = \mathbf{c}'(t) \cdot \nabla V(\mathbf{c}(t)) \quad \text{By chain rule.}$$
$$= \mathbf{F}(\mathbf{c}(t)) \cdot \nabla V(\mathbf{c}(t)) \quad \text{Since } \mathbf{c}(t) \text{ is a flow line.}$$
$$= -\nabla V(\mathbf{c}(t)) \cdot \nabla V(\mathbf{c}(t)) \quad \text{Since } \mathbf{F} = -\nabla V.$$
$$= -\|\nabla V(\mathbf{c}(t))\|^2 \quad \text{By definition of norm.}$$
$$\leq 0$$

Since the norm is always positive or zero, so it times -1 is always negative or zero.

Which shows that $\frac{d}{dt} V(\mathbf{c}(t)) \leq 0$, for any t. So $V(\mathbf{c}(t))$ is decreasing //

4.4.19. Calculate the scalar curl of the following vector field.

$$\mathbf{F}(x,y) = xy\,\hat{\imath} + (x^2 - y^2)\,\hat{\jmath}$$

Solution: Let $\mathbf{F}_1(x,y,z) = (\mathbf{F}(x,y), 0)$. Compute the curl of $\mathbf{F}_1$.

$$\text{curl } \mathbf{F}_1 = \nabla \times \mathbf{F}_1 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x^2-y^2 & 0 \end{vmatrix} = \hat{\imath}(0-0) - \hat{\jmath}(0-0) + \hat{k}\left(\frac{\partial}{\partial x}(x^2-y^2) - \frac{\partial}{\partial y}(xy)\right)$$
$$= \hat{k}(2x - x) = x\,\hat{k}.$$

Therefore, the scalar curl of $\mathbf{F}$ is x.

4.4.22 (a) For this question, let us compute the curl of each of the vector fields in Exercises 13-16.

For 13: $\mathbf{F}(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, could be a gradient field since $\text{curl}\,\mathbf{F} = \vec{0}$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{\imath}\left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right) - \hat{\jmath}\left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right) + \hat{k}\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right) = \hat{\imath}(0-0) - \hat{\jmath}(0-0) + \hat{k}(0-0)$$
$$= \vec{0}$$

or 14: $F(x,y,z) = yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k}$. Could be a gradient field since curl $F = \vec{0}$

url $F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \hat{\imath}\left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right) - \hat{\jmath}\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right) + \hat{k}\left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\right)$

$= \hat{\imath}(x - x) - \hat{\jmath}(y - y) + \hat{k}(z - z) = \vec{0}$ ✓

or 15: $F(x,y,z) = (x^2 + y^2 + z^2)(3\hat{\imath} + 4\hat{\jmath} + 5\hat{k})$. Let $a = x^2 + y^2 + z^2$.

rl $F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3a & 4a & 5a \end{vmatrix} = \hat{\imath}\left(\frac{\partial}{\partial y}(5a) - \frac{\partial}{\partial z}(4a)\right) - \hat{\jmath}\left(\frac{\partial}{\partial x}(5a) - \frac{\partial}{\partial z}(3a)\right) + \hat{k}\left(\frac{\partial}{\partial x}(4a) - \frac{\partial}{\partial y}(3a)\right)$

$= \hat{\imath}(10y - 8z) - \hat{\jmath}(10x - 6z) + \hat{k}(8x - 6y) \neq \vec{0}$, so ✓

cannot be a gradient field.

r 16: $F(x,y,z) = \dfrac{yz\hat{\imath} - xz\hat{\jmath} + xy\hat{k}}{x^2 + y^2 + z^2}$ Let $a = \dfrac{1}{x^2 + y^2 + z^2}$

l $F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ayz & -axz & axy \end{vmatrix} = \hat{\imath}\left(\frac{\partial}{\partial y}(axy) - \frac{\partial}{\partial z}(-axz)\right) - \hat{\jmath}\left(\frac{\partial}{\partial x}(axy) - \frac{\partial}{\partial z}(ayz)\right)$
$+ \hat{k}\left(\frac{\partial}{\partial x}(-axz) - \frac{\partial}{\partial y}(ayz)\right)$

$\hat{\imath}\left(\frac{\partial}{\partial y}(axy) + \frac{\partial}{\partial z}(axz)\right) - \hat{\jmath}\left(\frac{\partial}{\partial x}(axy) - \frac{\partial}{\partial z}(ayz)\right) - \hat{k}\left(\frac{\partial}{\partial x}(axz) + \frac{\partial}{\partial y}(ayz)\right)$

mpute each piece:

$\frac{\partial}{\partial y}(axy) = \frac{\partial}{\partial y}\left(\frac{xy}{x^2+y^2+z^2}\right) = \frac{x(x^2+y^2+z^2) - xy(2y)}{(x^2+y^2+z^2)^2} = \frac{x^3 - xy^2 + xz^2}{(x^2+y^2+z^2)^2}$

$\frac{\partial}{\partial z}(axz) = \frac{\partial}{\partial z}\left(\frac{xz}{x^2+y^2+z^2}\right) = \frac{x(x^2+y^2+z^2) - xz(2z)}{(x^2+y^2+z^2)^2} = \frac{x^3 - xz^2 + xy^2}{(x^2+y^2+z^2)^2}$

$\frac{\partial}{\partial x}(axy) = \frac{\partial}{\partial x}\left(\frac{xy}{x^2+y^2+z^2}\right) = \frac{y(x^2+y^2+z^2) - xy(2x)}{(x^2+y^2+z^2)^2} = \frac{y^3 - x^2y + yz^2}{(x^2+y^2+z^2)^2}$

$\frac{\partial}{\partial z}(ayz) = \frac{\partial}{\partial z}\left(\frac{yz}{x^2+y^2+z^2}\right) = \frac{y(x^2+y^2+z^2) - yz(2z)}{(x^2+y^2+z^2)^2} = \frac{y^3 - yz^2 + x^2y}{(x^2+y^2+z^2)^2}$

$\frac{\partial}{\partial x}(axz) = \frac{\partial}{\partial x}\left(\frac{xz}{x^2+y^2+z^2}\right) = \frac{z(x^2+y^2+z^2) - xz(2x)}{(x^2+y^2+z^2)^2} = \frac{z^3 - x^2z + y^2z}{(x^2+y^2+z^2)^2}$

$\frac{\partial}{\partial y}(ayz) = \frac{\partial}{\partial y}\left(\frac{yz}{x^2+y^2+z^2}\right) = \frac{z(x^2+y^2+z^2) - yz(2y)}{(x^2+y^2+z^2)^2} = \frac{z^3 - y^2z + x^2z}{(x^2+y^2+z^2)^2}$.

So, curl $F = \hat{\imath}\left(\dfrac{2x^3}{(x^2+y^2+z^2)^2}\right) - \hat{\jmath}\left(\dfrac{2yz^2+2x^2y}{(x^2+y^2+z^2)^2}\right) - \hat{k}\left(\dfrac{2z^3}{(x^2+y^2+z^2)^2}\right) \neq \vec{0}$

So F cannot be a gradient field.

4.4.22 (b) For this question, let us compute the divergence of each of the vector fields in Exercises 9–12.

For 9: $F(x,y) = x^3\hat{\imath} - x\sin(xy)\hat{\jmath}$

$\operatorname{div} F = \nabla \cdot F = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right)\cdot\left(x^3, -x\sin(xy)\right) = \dfrac{\partial}{\partial x}(x^3) - \dfrac{\partial}{\partial y}(x\sin(xy))$

$= 3x^2 - x^2\cos(xy) \neq 0$, so F cannot be the curl $\vec{V}$ for any $\vec{V}$

For 10: $F(x,y) = y\hat{\imath} - x\hat{\jmath}$.

$\operatorname{div} F = \nabla\cdot F = \dfrac{\partial}{\partial x}(y) + \dfrac{\partial}{\partial y}(-x) = 0+0=0$, so F could be the curl $\vec{V}$ of some vector field.

For 11: $F(x,y) = \sin(xy)\hat{\imath} - \cos(x^2y)\hat{\jmath}$

$\operatorname{div} F = \nabla\cdot F = \dfrac{\partial}{\partial x}(\sin(xy)) + \dfrac{\partial}{\partial y}(-\cos(x^2y)) = y\cos(xy) + x^2\sin(x^2y) \neq 0$, so

F cannot be the curl $\vec{V}$ for any vector field $\vec{V}$

For 12: $F(x,y) = xe^y\hat{\imath} - [y/(x+y)]\hat{\jmath}$.

$\operatorname{div} F = \nabla\cdot F = \dfrac{\partial}{\partial x}(xe^y) + \dfrac{\partial}{\partial y}\left(-\dfrac{y}{x+y}\right) = e^y - \left[\dfrac{x+y-y}{(x+y)^2}\right] = e^y - \dfrac{x}{(x+y)^2} \neq 0$

So F cannot be the curl $\vec{V}$ for any vector field $\vec{V}$.

4.4.27. Suppose $f,g,h:\mathbb{R}^2\to\mathbb{R}$ are differentiable. Show that the vector field $F(x,y,z) = (f(y,z), g(x,z), h(x,y))$ has zero divergence.

Pf: Let us compute the divergence of F.

$\operatorname{div} F = \nabla\cdot F = \dfrac{\partial}{\partial x}(f(y,z)) + \dfrac{\partial}{\partial y}(g(x,z)) + \dfrac{\partial}{\partial z}(h(x,y)) = 0+0+0=0$

Since we are differentiating constant functions with respect to the variable of differentiation

1.4.38. Let $\mathbf{r}(x,y,z) = (x,y,z)$ and $r = \sqrt{x^2+y^2+z^2} = \|\mathbf{r}\|$. Prove:

a). $\nabla(1/r) \overset{?}{=} -\mathbf{r}/r^3$, $r \neq 0$.

$$\nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \left(\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right), \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right), \frac{\partial}{\partial z}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\right)$$

$$= \left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}}\right) = -\frac{1}{(x^2+y^2+z^2)^{3/2}}(x,y,z)$$

$$= \frac{-\mathbf{r}}{(x^2+y^2+z^2)^{3/2}};\ \text{but } r^3 = (x^2+y^2+z^2)^{3/2}, \text{ so} \Rightarrow \boxed{\nabla(1/r) = \frac{-\mathbf{r}}{r^3}}$$

In general, Prove $\nabla(r^n) \overset{?}{=} n\, r^{n-2}\mathbf{r}$ and $\nabla(\log r) = \mathbf{r}/r^2$

$$\nabla(r^n) = \nabla\left((x^2+y^2+z^2)^{n/2}\right)$$

$$= \left(\frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{n/2}\right], \frac{\partial}{\partial y}\left[(x^2+y^2+z^2)^{n/2}\right], \frac{\partial}{\partial z}\left[(x^2+y^2+z^2)^{n/2}\right]\right)$$

$$= \left(\frac{n}{\cancel{2}} \cdot \cancel{2}x \cdot (x^2+y^2+z^2)^{\frac{n}{2}-1}, \frac{n}{\cancel{2}}\cancel{2}y(x^2+y^2+z^2)^{\frac{n}{2}-1}, \frac{n}{\cancel{2}}\cancel{2}z(x^2+y^2+z^2)^{\frac{n}{2}-1}\right)$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot (x,y,z) = n(x^2+y^2+z^2)^{\frac{n-2}{2}} \cdot \mathbf{r} = n \cdot r^{n-2}\mathbf{r}$$

$$\Rightarrow \boxed{\nabla(r^n) = n \cdot r^{n-2} \cdot \mathbf{r}}$$

Prove $\nabla(\log r) \overset{?}{=} \mathbf{r}/r^2$.

$$\nabla(\log r) = \nabla\left(\log\left(\sqrt{x^2+y^2+z^2}\right)\right) = \nabla\left(\frac{1}{2}\log(x^2+y^2+z^2)\right)$$

$$= \left(\frac{1}{2}\frac{\partial}{\partial x}\left(\log(x^2+y^2+z^2)\right), \frac{1}{2}\frac{\partial}{\partial y}\left(\log(x^2+y^2+z^2)\right), \frac{1}{2}\frac{\partial}{\partial z}\left(\log(x^2+y^2+z^2)\right)\right)$$

$$= \frac{1}{2}\left(\frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2}\right)$$

$$= \frac{1}{x^2+y^2+z^2}(x,y,z)$$

$$= \frac{\mathbf{r}}{r^2} \Rightarrow \boxed{\nabla(\log r) = \mathbf{r}/r^2}$$

M312 - Fall 2013 - HW2 - Enrique Areyan

4.4.38 (b). Prove that $\nabla^2(1/r) \stackrel{?}{=} 0$, $r \neq 0$.

By previous result and definition of ∇^2 we have:

$$\nabla^2(1/r) = \nabla \cdot \nabla(1/r) = \nabla\left(\frac{-\mathbf{r}}{r^3}\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{-y}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right)$$

We need only to compute one of these. All others are symmetrical. So,

$$\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{\cancel{2}} \frac{\cancel{2}x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}}$$

therefore,

$$\frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{x^2+y^2+z^2-3y^2}{(x^2+y^2+z^2)^{5/2}}; \quad \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{x^2+y^2+z^2-3z^2}{(x^2+y^2+z^2)^{5/2}}$$

Hence,

$$\nabla^2\left(\frac{1}{r}\right) = \frac{3x^2-x^2-y^2-z^2+3y^2-x^2-y^2-z^2+3z^2-x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}} = \frac{\cancel{3x^2}-\cancel{3x^2}+\cancel{3y^2}-\cancel{3y^2}+\cancel{3z^2}-\cancel{3z^2}}{(x^2+y^2+z^2)^{5/2}}$$

$$= 0. \Rightarrow \boxed{\nabla^2(1/r) = 0}$$

Prove that, in general $\nabla^2 r^n \stackrel{?}{=} n(n+1)r^{n-2}$

By previous result and definition of ∇^2 we have:

$$\nabla^2(r^n) = \nabla \cdot (\nabla(r^n)) = \nabla \cdot n r^{n-2}\mathbf{r} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot n r^{n-2}\mathbf{r}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(n(x^2+y^2+z^2)^{\frac{n-2}{2}}x, n(x^2+y^2+z^2)^{\frac{n-2}{2}}y, n(x^2+y^2+z^2)^{\frac{n-2}{2}}z\right)$$

$$= \frac{\partial}{\partial x}\left(n(x^2+y^2+z^2)^{\frac{n-2}{2}}x\right) + \frac{\partial}{\partial y}\left(n(x^2+y^2+z^2)^{\frac{n-2}{2}}y\right) + \frac{\partial}{\partial z}\left(n(x^2+y^2+z^2)^{\frac{n-2}{2}}z\right)$$

We need only to compute one of these. All others are symmetrical. So

$$\frac{\partial}{\partial x}\left(nx(x^2+y^2+z^2)^{\frac{n-2}{2}}\right) = n\left[(x^2+y^2+z^2)^{\frac{n-2}{2}} + \left(\frac{n-2}{\cancel{2}}\right)(\cancel{2}x^2)(x^2+y^2+z^2)^{\frac{n-2}{2}-1}\right]$$

$$= n(x^2+y^2+z^2)^{\frac{n-2}{2}}\left[1+(n-2)x^2(x^2+y^2+z^2)^{-1}\right]$$

So, $\frac{\partial}{\partial y}\left(ny(x^2+y^2+z^2)^{\frac{n-2}{2}}\right) = n(x^2+y^2+z^2)^{\frac{n-2}{2}}\left[1+(n-2)y^2(x^2+y^2+z^2)^{-1}\right]$

$$\frac{\partial}{\partial z}\left(nz(x^2+y^2+z^2)^{\frac{n-2}{2}}\right) = n(x^2+y^2+z^2)^{\frac{n-2}{2}}\left[1+(n-2)z^2(x^2+y^2+z^2)^{-1}\right]$$

Now we can compute the result:

$$\nabla^2(r^n) = n(x^2+y^2+z^2)^{\frac{n-2}{2}}\left[1+(n-2)x^2(x^2+y^2+z^2)^{-1}+1+(n-2)y^2(x^2+y^2+z^2)^{-1}+1+(n-2)z^2(x^2+y^2+z^2)^{-1}\right]$$

$$= n(x^2+y^2+z^2)^{\frac{n-2}{2}}\left[(x^2+y^2+z^2)^{-1}(n-2)\left[(x^2+y^2+z^2)\right]+3\right]$$

$$= n(x^2+y^2+z^2)^{\frac{n-2}{2}}\left[n-2+3\right] = n(n+1)(x^2+y^2+z^2)^{\frac{n-2}{2}} = n(n+1)r^{n-2}$$

$$\Rightarrow \boxed{\nabla^2(r^n) = n(n+1)r^{n-2}}$$

4.38 (c) Prove that $\nabla\cdot(\mathbf{r}/r^3) \stackrel{?}{=} 0$

$$\nabla\cdot(\mathbf{r}/r^3) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

Again, use symmetry and compute only one:

$$\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{\cancel{2}}\frac{\cancel{2}x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}}$$

Hence, $\frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{5/2}}$; $\frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$

Therefore,

$$\nabla\cdot(\mathbf{r}/r^3) = \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{5/2}} + \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}} = \frac{2y^2-2y^2+2x^2-2x^2+2z^2-2z^2}{(x^2+y^2+z^2)^{5/2}} = 0$$

$$\Rightarrow \boxed{\nabla\cdot(\mathbf{r}/r^3) = 0}$$

M312 - Fall 2013 - HW2 - Enrique Areyan

In general, prove $\nabla\cdot(r^n\mathbf{r}) \stackrel{?}{=} (n+3)r^n$

$$\nabla\cdot(r^n\mathbf{r}) = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot\left((x^2+y^2+z^2)^{n/2}x,(x^2+y^2+z^2)^{n/2}y,(x^2+y^2+z^2)^{n/2}z\right)$$

$$= \frac{\partial}{\partial x}\left(x(x^2+y^2+z^2)^{n/2}\right)+\frac{\partial}{\partial y}\left(y(x^2+y^2+z^2)^{n/2}\right)+\frac{\partial}{\partial z}\left(z(x^2+y^2+z^2)^{n/2}\right)$$

Again; symmetry:

$$\frac{\partial}{\partial x}\left(x(x^2+y^2+z^2)^{n/2}\right) = (x^2+y^2+z^2)^{n/2}+\frac{n}{\cancel{2}}(\cancel{2}x^2)(x^2+y^2+z^2)^{\frac{n}{2}-1}$$
$$= (x^2+y^2+z^2)^{n/2}\left(1+nx^2(x^2+y^2+z^2)^{-1}\right)$$

$$\frac{\partial}{\partial y}\left(y(x^2+y^2+z^2)^{n/2}\right) = (x^2+y^2+z^2)^{n/2}\left(1+ny^2(x^2+y^2+z^2)^{-1}\right)$$

$$\frac{\partial}{\partial z}\left(z(x^2+y^2+z^2)^{n/2}\right) = (x^2+y^2+z^2)^{n/2}\left(1+nz^2(x^2+y^2+z^2)^{-1}\right)$$

$$\nabla\cdot(r^n r) = (x^2+y^2+z^2)^{n/2}\left[1+nx^2(x^2+y^2+z^2)^{-1}+1+ny^2(x^2+y^2+z^2)+1+nz^2(x^2+y^2+z^2)^{-1}\right]$$
$$= (x^2+y^2+z^2)^{n/2}\left[(x^2+y^2+z^2)^{-1}(n(x^2+y^2+z^2))+3\right]$$
$$= (x^2+y^2+z^2)^{n/2}[n+3]$$
$$= (n+3)r^n \implies \boxed{\nabla\cdot(r^n\mathbf{r}) = (n+3)r^n}$$

4.4.38 (d) Prove that $\nabla\times\mathbf{r}\stackrel{?}{=}\mathbf{0}$

$$\nabla\times\mathbf{r} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix} = \hat{\imath}\left(\frac{\partial}{\partial y}(z)-\frac{\partial}{\partial z}(y)\right)-\hat{\jmath}\left(\frac{\partial}{\partial x}(z)-\frac{\partial}{\partial z}(x)\right)+\hat{k}\left(\frac{\partial}{\partial x}(y)-\frac{\partial}{\partial y}(x)\right)$$
$$= \hat{\imath}(0-0)-\hat{\jmath}(0-0)+\hat{k}(0-0) = \mathbf{0} \implies \boxed{\nabla\times\mathbf{r}=\mathbf{0}}$$

Prove in general $\nabla\times(r^n\mathbf{r})\stackrel{?}{=}\mathbf{0}$

$$\nabla\times(r^n\mathbf{r}) = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ r^nx & r^ny & r^nz\end{vmatrix} = \hat{\imath}\left(\frac{\partial}{\partial y}(r^nz)-\frac{\partial}{\partial z}(r^ny)\right)-\hat{\jmath}\left(\frac{\partial}{\partial x}(r^nz)-\frac{\partial}{\partial z}(r^nx)\right)+\hat{k}\left(\frac{\partial}{\partial x}(r^ny)-\frac{\partial}{\partial y}(r^nx)\right)$$

Compute each piece:

$$\frac{\partial}{\partial y}(r^n z) = \frac{\partial}{\partial y}\left((x^2+y^2+z^2)^{\frac{n}{2}} z\right) = nyz(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$$\frac{\partial}{\partial z}(r^n y) = \frac{\partial}{\partial z}\left((x^2+y^2+z^2)^{\frac{n}{2}} y\right) = nyz(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$$\frac{\partial}{\partial x}(r^n z) = \frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{\frac{n}{2}} z\right) = nxz(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$$\frac{\partial}{\partial z}(r^n x) = \frac{\partial}{\partial z}\left((x^2+y^2+z^2)^{\frac{n}{2}} x\right) = nxz(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$$\frac{\partial}{\partial x}(r^n y) = \frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{\frac{n}{2}} y\right) = nxy(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$$\frac{\partial}{\partial y}(r^n x) = \frac{\partial}{\partial y}\left((x^2+y^2+z^2)^{\frac{n}{2}} x\right) = nxy(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

herefore,

$$\nabla \times (r^n \mathbf{r}) = \hat{i}\left(nyz(x^2+y^2+z^2)^{\frac{n}{2}-1} - nyz(x^2+y^2+z^2)^{\frac{n}{2}-1}\right) - \hat{j}\left(nxz(x^2+y^2+z^2)^{\frac{n}{2}-1} - nxz(x^2+y^2+z^2)^{\frac{n}{2}-1}\right)$$
$$+ \hat{k}\left(nxy(x^2+y^2+z^2)^{\frac{n}{2}-1} - nxy(x^2+y^2+z^2)^{\frac{n}{2}-1}\right) = 0$$

$$\Rightarrow \boxed{\nabla \times (r^n \mathbf{r}) = 0}$$