M312 - HW 1 - Enrique Areyan - Fall 2013. ①

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(4.1.12) Let $\mathbf{v}$ and $\mathbf{a}$ denote the velocity and acceleration vectors of a particle moving on a path $\mathbf{c}$. Suppose the initial position of the particle is $\mathbf{c}(0)=\langle 3,4,0\rangle$, the initial velocity is $\mathbf{v}(0)=\langle 1,1,-2\rangle$, and the acceleration function is $\mathbf{a}(t)=\langle 0,0,6\rangle$. Find $\mathbf{v}(t)$ and $\mathbf{c}(t)$.

Solution:

By definition $\mathbf{v}(t)=\int \mathbf{a}(t)dt=\langle \int 0dt, \int 0dt, \int 6dt\rangle$
$=\langle c_1, c_2, 6t+c_3\rangle$, for constants $c_1,c_2,c_3\in\mathbb{R}$

To find the constants we use the initial condition:

$$\mathbf{v}(0)=\langle 1,1,-2\rangle=\langle c_1,c_2,6\cdot 0+c_3\rangle=\langle c_1,c_2,c_3\rangle$$

Hence, $c_1=1=c_2$ and $c_3=-2$. the velocity is given by:

$$\boxed{\mathbf{v}(t)=\langle 1,1,6t-2\rangle}$$

Likewise, for the position: $\mathbf{c}(t)=\int \mathbf{v}(t)dt=\langle \int dt, \int dt, \int 6t-2dt\rangle$,
$=\langle t+d_1, t+d_2, 3t^2-2t+d_3\rangle$,
for constants $d_1,d_2,d_3\in\mathbb{R}$

To find the constants we use the initial condition:

$$\mathbf{c}(0)=\langle 3,4,0\rangle=\langle 0+d_1, 0+d_2, 3(0)^2-2(0)+d_3\rangle$$
$$=\langle d_1,d_2,d_3\rangle$$

Hence, $d_1=3, d_2=4, d_3=0$. the position is given by

$$\boxed{\mathbf{c}(t)=\langle t+3, t+4, 3t^2-2t\rangle}$$

1.1.13) the acceleration, initial velocity, and initial position of a particle
raveling through space are given by:

$$a(t)=\langle 2,-6,-4\rangle,\quad v(0)=\langle -5,1,3\rangle,\quad r(0)=\langle 6,-2,1\rangle.$$

he particle's trajectory intersects the yz plane exactly twice.
ind these intersection points.

olution: First, recover the position function $r(t)$ as in (4.1.12).

By definition $v(t)=\int a(t)\,dt=\langle \int 2\,dt, \int -6\,dt, \int -4\,dt\rangle$
$=\langle 2t+C_1, -6t+C_2, -4t+C_3\rangle$
for constants $C_1, C_2, C_3 \in \mathbb{R}$.

Find constants: $v(0)=\langle -5,1,3\rangle=\langle C_1, C_2, C_3\rangle$. Hence, $C_1=-5$, $C_2=1$, $C_3=3$.

the velocity is given by $\boxed{v(t)=\langle 2t-5, -6t+1, -4t+3\rangle}$

kewise, for the position: $r(t)=\int v(t)\,dt=\langle \int 2t-5\,dt, \int -6t+1\,dt, \int -4t+3\,dt\rangle$
$=\langle t^2-5t+d_1, -3t^2+t+d_2, -2t^2+3t+d_3\rangle$
for constants $d_1, d_2, d_3 \in \mathbb{R}$.

nd constants: $r(0)=\langle 6,-2,1\rangle=\langle d_1, d_2, d_3\rangle$ Hence, $d_1=6$, $d_2=-2$, $d_3=1$.

e position is given by $\boxed{r(t)=\langle t^2-5t+6, -3t^2+t-2, -2t^2+3t+1\rangle}$

w, points in the yz plane have the form $(0, y_0, z_0)$.
find the intersection points of the particle with this plane we find t
:

$$r(t)=\langle t^2-5t+6, -3t^2+t-2, -2t^2+3t+1\rangle=\langle 0, y_0, z_0\rangle.$$

$$\Rightarrow\ t^2-5t+6=0 \Rightarrow (t-3)(t-2)=0 \Rightarrow t=3 \text{ or } t=2.$$

So the two points of intersection are:

$$r(3)=\langle 9-15+6, -27+3-2, -18+9+1\rangle=\boxed{(0,-26,-8)}$$

$$r(2)=\langle 4-10+6, -12+2-2, -8+6+1\rangle=\boxed{(0,-12,-1)}$$

(4.1.15) If $r(t) = 6t\hat{\imath} + 3t^2\hat{\jmath} + t^3\hat{k}$, what force acts on a particle of mass m moving along r at $t=0$?

Solution: By Newton's Second Law: $F = ma$

So we need to obtain the acceleration in order to obtain the force:

$a(t) = r''(t) = \langle 0, 6, 6t \rangle$. Hence,

$F = ma = m \cdot \langle 0, 6, 6t \rangle$. In particular, at $t=0$ the force is:

$$\boxed{F(t=0) = m\langle 0, 6, 0 \rangle}$$

(4.1.24) Let c be a path in $\mathbb{R}^3$ with zero acceleration. Prove that c is a straight line or a point.

Solution: Let $a(t)$ be the acceleration of a path such that $a(t) = \mathbf{0} = \langle 0, 0, 0 \rangle$. To obtain the curve c, we integrate twice:

$$c(t) = \int \left[\int a(t)\,dt \right] dt = \int \left[\langle \int 0\,dt, \int 0\,dt, \int 0\,dt \rangle \right] dt$$

$$= \int \langle c_1, c_2, c_3 \rangle dt \quad \text{, for constants } c_1, c_2, c_3 \in$$

$$= \langle c_1 t + d_1, c_2 t + d_2, c_3 t + d_3 \rangle, \quad \text{constants } d_1, d_2, d_3$$

$$\Rightarrow \boxed{c(t) = \langle c_1 t + d_1, c_2 t + d_2, c_3 t + d_3 \rangle}$$

Depending on the values of the constant there are two cases:

(i) $c_1 = c_2 = c_3 = 0$. In this case $c(t) = \langle d_1, d_2, d_3 \rangle$ is a point

OR

(ii) at least one of c_1, c_2, c_3 is not zero. In this case we get a line since at least one coordinate is a linear functions with the others being linear functions or constants

4.2.3) Find the arc length of the given curve on the specified interval

$(\sin 3t, \cos 3t, 2t^{3/2})$, for $0 \leq t \leq 1$

Solution: By definition the arc length:

$$L = \int_0^1 \sqrt{(x(t)')^2 + (y(t)')^2 + (z(t)')^2}\, dt$$

$$= \int_0^1 \sqrt{(\sin(3t)')^2 + (\cos(3t)')^2 + [(2t^{3/2})']^2}\, dt$$

$$= \int_0^1 \sqrt{9(\cos^2(3t) + \sin^2(3t)) + 9t}\, dt = \int_0^1 \sqrt{9+9t}\, dt$$

$= 3\int_0^1 \sqrt{1+t}\, dt$. Make the change $u = 1+t \Rightarrow du = dt$.
(ignore limits for now)

$\rightarrow 3\int \sqrt{u}\, du = 3\left(\frac{2}{3}u^{3/2}\right) = 2u^{3/2}$. Substitute back:

$$2\left[u^{3/2}\right] = 2\left[(1+t)^{3/2}\right]_0^1 = 2(2^{3/2} - 1) = \boxed{2(2\sqrt{2} - 1)}$$

2.5) Find the arc length of the given curve on the specified interval.

(t, t, t^2), for $1 \leq t \leq 2$

Solution: By definition of arc length

$$L = \int_1^2 \sqrt{(t')^2 + (t')^2 + (t^{2\prime})^2}\, dt = \int_1^2 \sqrt{2+4t^2} = 2\int_1^2 \sqrt{\tfrac{1}{2} + t^2}\, dt$$

$$2\left[\frac{t}{2}\sqrt{\tfrac{1}{2} + t^2} + \frac{\frac{1}{2}}{2}\ln\left(t + \sqrt{\tfrac{1}{2} + t^2}\right)\right]_1^2 = \left[t\sqrt{\frac{2t^2+1}{2}} + \frac{1}{2}\ln\left(t + \sqrt{\frac{2t^2+1}{2}}\right)\right]_1^2$$

$$\left(2\sqrt{\tfrac{9}{2}} + \tfrac{1}{2}\ln\left(2 + \sqrt{\tfrac{9}{2}}\right)\right) - \left(\sqrt{\tfrac{3}{2}} + \tfrac{1}{2}\ln\left(1 + \sqrt{\tfrac{3}{2}}\right)\right)$$

$$\frac{6}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{2}\ln\left[\frac{\frac{2\sqrt{2}+3}{\sqrt{2}}}{\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}}}\right] = \boxed{\frac{6-\sqrt{3}}{\sqrt{2}} + \frac{1}{2}\ln\left(\frac{2\sqrt{2}+3}{\sqrt{2}+\sqrt{3}}\right)}$$

(4.2.7) Find the arc length of $c(t)=(t,|t|)$, for $-1\leq t\leq 1$.

Solution: We can divide these curve into two pieces:

$c(t)=c_1(t)\cup c_2(t)$, where $c_1(t)=(t,-t)$ for $-1\leq t\leq 0$
Since the function $|-t|$ $c_2(t)=(t,t)$ for $0\leq t\leq 1$.
does not have a derivative at $t=0$.

Hence, lenght of c = lenght of c_1 + lenght of c_2.

$$L(c_1)=\int_{-1}^{0}\sqrt{(t')^2+(-t')^2}\,dt=\int_{-1}^{0}\sqrt{2}\,dt=\sqrt{2}\,t\Big|_{-1}^{0}=\sqrt{2}$$

$$L(c_2)=\int_{0}^{1}\sqrt{(t')^2+(t')^2}\,dt=\int_{0}^{1}\sqrt{2}\,dt=\sqrt{2}\,t\Big|_{0}^{1}=\sqrt{2}$$

therefore, the lenght of c is $\boxed{2\sqrt{2}}$

(4.2.8) Let $c(t)=(Rt-R\sin t,R-R\cos t)$ for $0\leq t\leq 2\pi$, be a paramet[rization] of one arch of the cycloid. then.

$$L(c)=\int_{0}^{2\pi}\sqrt{[(Rt-R\sin t)']^2+[(R-R\cos t)']^2}\,dt=\int_{0}^{2\pi}\sqrt{(R-R\cos t)^2+(R\sin t)^2}\,dt$$

$$=\int_{0}^{2\pi}\sqrt{R^2-2R^2\cos t+R^2\cos^2 t+R^2\sin^2 t}\,dt=\int_{0}^{2\pi}\sqrt{2R^2-2R^2\cos t}\,dt$$

$$=\sqrt{2}R\int_{0}^{2\pi}\sqrt{1-\cos t}\,dt = \text{by double angle formula} = 2R\int_{0}^{2\pi}\left|\sin\left(\frac{t}{2}\right)\right|dt$$

$$=4R\left[-\cos\left(\frac{t}{2}\right)\right]_{0}^{2\pi}=4R[-\cos(\pi)+\cos(0)]=4R[1+1]=8R=4(2R),$$

where the diameter is $2R$, thus showing the result.

9) Compute the lenght of the hypocycloid

$$c(t) = \langle \sin^3 t, \cos^3 t \rangle; \text{ for } 0 \leq t \leq 2\pi.$$

olution:

$$L(c) = \int_0^{2\pi} \sqrt{(\sin^3 t')^2 + (\cos^3 t')^2}\, dt = \int_0^{2\pi} \sqrt{(3\sin^2 t \cos t)^2 + (-3\cos^2 t \sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9(\sin^4 t \cos^2 t + \sin^2 t \cos^4 t)}\, dt$$

$$= 3\int_0^{2\pi} \sqrt{\sin^2 t \cos^2 t\,(\sin^2 t + \cos^2 t)}\, dt$$

$$= 3\int_0^{2\pi} \sqrt{\sin^2 t \cos^2 t}\, dt = 3\int_0^{2\pi} |\sin t \cos t|\, dt = \text{by double angle formula}$$

$$= \frac{3}{2}\int_0^{2\pi} |\sin(2t)|\, dt$$ .. Note that we are taking absolute value of $\sin(2t)$ so we must analyze its behavior:

$\sin(2t) \geq 0$ if $0 \leq t \leq \pi/2$ or $\pi \leq t \leq 3\pi/2$

$\sin(2t) \leq 0$ if $\pi/2 \leq t \leq \pi$ or $\frac{3\pi}{2} \leq t \leq 2\pi$

us compute the arclenght for one of these intervals:

$$\frac{3}{2}\int_0^{\pi/2} |\sin(2t)|\, dt = \frac{3}{4}\Big[-\cos(2t)\Big]_0^{\pi/2} = \frac{3}{4}\left[-\cos(\pi) + \cos(0)\right] = \frac{3}{4}[1+1] = \frac{6}{4}$$

is is the same result for all four integrals (since we are taking absolute value)

erefore, the arc lenght is:

$$(4)\left(\frac{3}{2}\right)\int_0^{\pi/2} |\sin(2t)|\, dt = (4)\left(\frac{3}{4}\right)[1+1] = \boxed{6}$$

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(4.2.16) Let $c:[a,b]\to\mathbb{R}^3$ be an infinitely differentiable path. Assume $c'(t)\neq 0$ for any t. the vector $T(t)=\frac{c'(t)}{\|c'(t)\|}$ is tangent to c at $c(t)$.

(a) SHOW that $T'(t)\cdot T(t)=0$.

Pf: Since $T(t)$ is a unit vector, i.e., $\|T(t)\|=1$, we know that

$T(t)\cdot T(t)=1$. Now, differentiate both sides

$\frac{d}{dt}(T(t)\cdot T(t))=\frac{d}{dt}(1)$ By the product rule

$2T'(t)\cdot T(t)=0 \iff \boxed{T'(t)\cdot T(t)=0}$ $T'(t)\perp T(t)$.

(b) Write down a formula for $T'(t)$ in terms of c.

Solution: Note that $T(t)=\frac{c'(t)}{\|c'(t)\|}=\frac{c'(t)}{\sqrt{c'(t)\cdot c'(t)}}$

Now, compute:

$$T'(t)=\frac{d}{dt}(T(t))=\frac{d}{dt}\left(\frac{c'(t)}{\sqrt{c'(t)\cdot c'(t)}}\right)=\frac{c''(t)\sqrt{c'(t)\cdot c'(t)}-c'(t)(\sqrt{c'(t)\cdot c'(t)})'}{(\sqrt{c'(t)\cdot c'(t)})^2}$$

$$=\frac{c''(t)\sqrt{c'(t)\cdot c'(t)}-c'(t)\frac{\cancel{2}c''(t)\cdot c'(t)}{\cancel{2}\sqrt{c'(t)c'(t)}}}{c'(t)\cdot c'(t)}=\frac{c''(t)\sqrt{c'(t)\cdot c'(t)}-c'(t)\frac{c''(t)\cdot c'(t)}{\sqrt{c'(t)\cdot c'(t)}}}{c'(t)\cdot c'(t)}$$

$$=\frac{\frac{c''(t)(c'(t)\cdot c'(t))-c'(t)(c''(t)\cdot c'(t))}{\sqrt{c'(t)\cdot c'(t)}}}{c'(t)\cdot c'(t)}=\boxed{\frac{c''(t)(c'(t)\cdot c'(t))-c'(t)(c''(t)\cdot c'(t))}{(c'(t)\cdot c'(t))^{3/2}}}$$