Calculus IV - Enrique Areyan - Fall 2013

Vector-valued functions

$\mathbf{c}: \mathbb{R} \to \mathbb{R}^n$ OR $\mathbf{c}: (a,b) \to \mathbb{R}^n$ (usually $n=2$ OR $n=3$).

$\mathbf{c}(t) = \langle x_1(t), x_2(t), \ldots, x_n(t) \rangle$. $\mathbf{c}'(t) = \langle x_1'(t), x_2'(t), \ldots, x_n'(t) \rangle$ is a tangent vector to $\mathbf{c}(t)$. (velocity vect

$\|\mathbf{c}'(t)\| =$ speed.

BASIC LAWS:

$\frac{d}{dt}(\mathbf{b}(t) \pm \mathbf{c}(t)) = \mathbf{b}'(t) \pm \mathbf{c}'(t)$; $\frac{d}{dt}(p(t)\mathbf{c}(t)) = p'(t)\mathbf{c}(t) + p(t)\mathbf{c}'(t)$

$\frac{d}{dt}(\mathbf{b}(t) \cdot \mathbf{c}(t)) = \mathbf{b}'(t)\cdot\mathbf{c}(t) + \mathbf{b}(t)\cdot\mathbf{c}'(t)$; $\frac{d}{dt}(\mathbf{b}(t) \times \mathbf{c}(t)) = \mathbf{b}'(t)\times\mathbf{c}(t) + \mathbf{b}(t)\times\mathbf{c}'(t)$

$\frac{d}{dt}(\mathbf{c}(q(t))) = q'(t)\,\mathbf{c}'(q(t))$

$\mathbb{R} \xrightarrow{\mathbf{c}} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$ where $\mathbf{c}(t) = (x_1(t), \ldots, x_n(t))$; then

$$\frac{d}{dt}(f(\mathbf{c}(t))) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \frac{\partial f}{\partial x_1}(\mathbf{c}(t))x_1'(t) + \frac{\partial f}{\partial x_2}(\mathbf{c}(t))x_2'(t) + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{c}(t))x_n'(t)$$

SOME DEFINITIONS:

A function f is in $\underline{C^1}$ if it is derivable and its derivative is continuous.

Note that If $\mathbf{c}(t)$ is a C^1 function then the image does not have to be smoo

Def: we say that a path $\mathbf{c}(t)$ is regular at t_0 if $\mathbf{c}'(t_0) \neq 0$.

we say that a path $\mathbf{c}(t)$ is regular if $\mathbf{c}'(t) \neq 0$ for all t.

If $\mathbf{c}(t)$ is a regular path then $\mathbf{c}(t)$ traces out a smooth curve.

Newton's second law: a particle of mass m traveling along a path $\mathbf{c}(t)$ that is acted by a force $\mathbf{F}$ must satisfy $\mathbf{F}(\mathbf{c}(t)) = m\mathbf{a}(t)$.

$\mathbf{a}(t) = \mathbf{c}''(t)$ is the acceleration of path $\mathbf{c}(t)$.

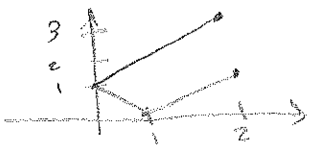
Arc Length: the arc length of a path $\mathbf{c}(t)$ for $t_0 \leq t \leq t_1$ is defined to be

$$L(\mathbf{c}) = \int_{t_0}^{t_1} \|\mathbf{c}'(t)\|\,dt = \int_{t_0}^{t_1} \sqrt{(x_1'(t))^2 + (x_2'(t))^2 + \cdots + (x_n'(t))^2}\,dt$$

Arc Length is independent of parametrization.

Def: A curve $\mathbf{c}: [a,b] \to \mathbb{R}^n$ is called piecewise C^1 if $\mathbf{c}$ is continuous and there is a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that $\mathbf{c}$ is C^1 on $[t_j, t_{j-}$ for every $j = 1, 2, \ldots, n$.

Example:



$\mathbf{c}(t) = (|t|, |t-1|)$

$-2 \leq t \leq 2$

Note that if a path $\mathbf{c}(t)$ is not smooth but piecewise smooth, we can find the arc length of $\mathbf{c}(t)$ by adding the arcs lengths of the pieces.

VECTOR FIELDS:

Def: A vector field is a function $\mathbf{F}: \mathbb{R}^n \to \mathbb{R}^n$, or from a subset $\mathbf{F}: A \subseteq \mathbb{R}^n \to \mathbb{R}^n$
A scalar field is a function $f: \mathbb{R}^n \to \mathbb{R}$, or from a subset $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$.

Def: for any differentiable function $f: \mathbb{R}^n \to \mathbb{R}$ its gradient defines a vector field $\nabla f: \mathbb{R}^n \to \mathbb{R}^n$ called a gradient field $\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$.

Not all vectors fields are gradient fields.

Properties: ∇f points in the direction along which f increases the fastest.
∇f is perpendicular to the level surfaces of f.

Pf: Let $c(t)$ = curve in a level surface. then $f(\mathbf{c}(t)) = C_0$. derive: $\nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = 0$).

Def: A path $\mathbf{c}(t)$ is a flow line for a vector field $\mathbf{F}$ if $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$

Geometrically, $\mathbf{c}(t)$ is a flow line for $\mathbf{F}$ if $\mathbf{c}(t)$ is tangent to the vectors on $\mathbf{F}$ at every point, lying on the curve traced out by $\mathbf{c}(t)$.

Example: $F(x,y) = (-y, x)$

Divergence:

Def: the del operator in $\mathbb{R}^n$ is defined to be $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right)$

Def: the divergence of a vector field $\mathbf{F} = (F_1, F_2, \dots, F_n)$ is $\boxed{\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}}$

$$\boxed{\operatorname{div} F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \dots + \frac{\partial F_n}{\partial x_n}}$$

Interpretation: $\operatorname{div} \mathbf{F}$ represents the rate of expansion per unit volume (or area) under the flow of the gas (if we imagine $\mathbf{F}$ to be the velocity of a gas)

If $\operatorname{div} \mathbf{F} < 0$ then the gas is compressing. If $\operatorname{div} \mathbf{F} > 0$ then the gas is expanding

Curl:

Def: the curl of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is $\boxed{\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}}$

$$\boxed{\operatorname{curl} \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)}$$

Interpretation: $\operatorname{curl} \mathbf{F}$ measures the tendency for the vector field to swirl around

A vector field $\mathbf{F}$ is called <u>incompressible</u> if $\text{div}\,\mathbf{F} = 0$ (neither diverging nor compre

A vector field $\mathbf{F}$ in $\mathbb{R}^3$ is called <u>irrotational</u> if $\text{curl}\,\mathbf{F} = 0$ (it is not rotating).

<u>SCALAR curl</u> if $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$; we can define $\mathbf{F}_1 = (F_1, F_2, 0)$, where $\mathbf{F} = (F_1, F_2)$.
Computing like the curl we get that Scalar curl $\mathbf{F} = \text{curl}\,\mathbf{F}_1 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

<u>THEOREM 1</u>: Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a C^2 scalar function. then $\boxed{\text{curl}\,\nabla f = 0}$
therefore, to show that a given vector field $\mathbf{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is <u>not a gradient</u> field, we need only show that $\text{curl}\,\mathbf{F} \neq 0$.

<u>THEOREM 2</u>: Let $\mathbf{F}: \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2, 3-dimensional vector field. then $\text{div}(\text{curl}\,\mathbf{F}) =$
therefore, to show that a given vector field $\mathbf{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is <u>not a curl</u>, we need only to show that $\text{div}\,\mathbf{F} \neq 0$.

<u>Def</u>: the <u>Laplace operator</u> $\Delta = \nabla^2$ is defined to be the divergence of the gradient. $\nabla^2 f = \Delta f = \text{div}(\nabla f) = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

(For basic Identities of Vector Analysis, look at page 255).

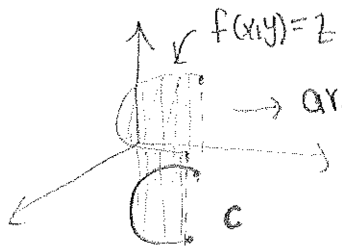
<u>PATH INTEGRAL</u>: we are given a scalar function $f: \mathbb{R}^3 \to \mathbb{R}$, and its integral along the path $\mathbf{c}$, is defined when $\mathbf{c}: I = [a,b] \to \mathbb{R}^3$ is of class C^1, with $\mathbf{c}(t) = (x(t), y(t), z(t))$. like this:

$$\boxed{\int_c f\,ds = \int_a^b f(\mathbf{c}(t))\,\|\mathbf{c}'(t)\|\,dt}$$

<u>Remark</u>: If $f \equiv 1$ then $\int_c f\,ds = \int_a^b \|\mathbf{c}'(t)\|\,dt = L(\mathbf{c})$

<u>Remark 2</u>: If $\mathbf{c}(t)$ is only piecewise C^1 or $f(\mathbf{c}(t))$ is piecewise continuous, define $\int_c f\,ds$ by breaking $[a,b]$ into pieces over which $f(\mathbf{c}(t))\|\mathbf{c}'(t)\|$ is continuou

<u>Geometric Interpretation for planar curves</u>: $\mathbf{c}(t): \underset{[a,b]}{\mathbb{R}} \to \mathbb{R}^2$ and $f: \mathbb{R}^2 \to \mathbb{R}$; then $\int_c f\,ds = \int_a^b f(\mathbf{c}(t))\|\mathbf{c}'(t)\|\,dt$ (If $f(x,y) \geq 0$) is the area of the fence constructed with base the image of $\mathbf{c}$ and with height $f(x,y)$ at (



$\to$ area of fence $\int_c f\,ds$.

urvature of a curve: curvature of a line = 0. curvature of a circle = $\frac{1}{r}$, where is the radius of the circle. Big radius → small curvature (think of the earth). mall radius → big curvature (think of driving and making a U-turn).

ef: $c:[a,b]\to\mathbb{R}^n$, $c'(t)\neq 0$ for any t, we say that $c(t)$ is parametrized by arc ength if $\|c'(t)\|=1$ for all t. ($\int_a^b \|c'(t)\|dt = b-a$).

ef: If $c(t)$ is parametrized by arc-length $\|c'(t)\|=1$, then $K(p)=\|c''(t)\|$, here K is the curvature at $p\in C$, $p=c(t)$.

dimension 3, this can be rewritten as: $K(t)=\dfrac{\|c'(t)\times c''(t)\|}{\|c'(t)\|^3}$

ef: the total curvature is $\int_C K ds = \int_a^b K(t)\|c'(t)\|dt$

NE INTEGRALS: Let F be a vector field on $\mathbb{R}^3$ that is continuous on C path $c:[a,b]\to\mathbb{R}^3$. We define the line integral of F along c by

$$\int_C F\cdot ds = \int_a^b F(c(t))\cdot c'(t)dt$$

Remark 1: the integrand is a dot product, so it is actually a scalar.

emark 2: as with path integrals, if $F(c(t))\cdot c'(t)$ is only piecewise continuous e can compute the line integral by breaking c into pieces.

ernative formula: $\int_C F\cdot ds = \int_a^b [F(c(t))\cdot T(t)]\|c'(t)\|dt$; where $T(t)=\frac{c'(t)}{\|c'(t)\|}$ if $c'(t)\neq 0$.

e that this is a path integral over $f=F(c(t))\cdot T(t)$

terpretation: Recall that work is defined by $W=F\cdot d$, F = force, d = displacement. usual break the curve into infinestimal pieces to compute the work done by a ticle moving along the path $c:[a,b]\to\mathbb{R}^3$ in a force field F: $W=\int_a^b F(c(t))\cdot c'(t)dt$.

ferential form: (differential notation) an alternative way of writing line integrals: (F_1,F_2,F_3):

$$\int_C F_1dx+F_2dy+F_3dz = \int_a^b \left(F_1\frac{dx}{dt}+F_2\frac{dy}{dt}+F_3\frac{dz}{dt}\right)dt = \int_C F\cdot ds$$

arametrization:

: given $c:[a,b]\to\mathbb{R}^n$ a piecewise C^1 path. A reparametrization of c is $p=c\circ h:[a_1,b_1]\to\mathbb{R}^n$, where $h:[a_1,b_1]\to[a,b]$ is a 1-1 and onto map and $h'(s)\neq 0$.

ark: this definition imply that h has to be strictly increasing or strictly decreasing map end points to end points.

ark 2: It is called a reparametrization because the image of the curve will be same.

ARK 3: We can always find reparametrization with constant speed $\|c'(t)\|=1$ = arc-length parametrization

Two possibilities:

1. $h' > 0$: $h(a_1) = a$, $h(b_1) = b$ $\Rightarrow$ orientation-preserving reparametrization.
2. $h' < 0$: $h(a_1) = b$, $h(b_1) = a$ $\Rightarrow$ orientation-reversing reparametrization.

theorem: Let $\mathbf{F}$ be a continuous vector field on a path $\mathbf{c}$ and $\mathbf{p}$ a reparametrization of
If $\mathbf{p}$ is orientation preserving then $\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{p}} \mathbf{F}\cdot d\mathbf{s}$ (Pf follows from change of variable and chain rule).
If $\mathbf{p}$ is orientation reversing then $\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = -\int_{\mathbf{p}} \mathbf{F}\cdot d\mathbf{s}$

Remark: substitute end points to check the respective orientations.

theorem: Let f be a continuous scalar function on a path $\mathbf{c}$ and $\mathbf{p}$ a reparametrization of
then: $\int_{\mathbf{c}} f\,ds = \int_{\mathbf{p}} f\,ds$ (the path integral is not an oriented integral).
(Pf follows from change of variable).

THEOREM: Let $\mathbf{c}:[a,b]\to\mathbb{R}^n$ be a piecewise C^1 path and f a scalar C^1 function on $\mathbf{c}$.
then: $\boxed{\int_{\mathbf{c}} \nabla f\cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))}$

Pf: $\int_{\mathbf{c}} \nabla f\cdot d\mathbf{s} = \int_a^b \nabla f(\mathbf{c}(t))\cdot \mathbf{c}'(t)\,dt = \int_a^b \frac{d}{dt}(f(\mathbf{c}(t)))\,dt = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$.

STRATEGY: If we cannot compute a line integral as given, we can try to see if $\mathbf{F} = \nabla$
If so, compute f and use above theorem.

THEOREM: if $\mathbf{F} = (F_1, F_2)$ is a C^1 vector field on $\mathbb{R}^2$, then the following conditions are equivalents: i) $\mathbf{F}$ is a gradient field, ii) curl $\mathbf{F} = 0$, iii) $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$

Def: A simple curve C is the image of a piecewise C^1 map $\mathbf{c}:[a,b]\to\mathbb{R}^3$ which is one-to-one on $[a,b]$ (no intersection like ℓ).
If we specify P, Q to be the endpoints of C, then there are two possible orientations on C: either from P to Q or from Q to P and we call the curve an oriented simple curve.

Def: A closed simple curve is a simple curve such that $\mathbf{c}(a) = \mathbf{c}(b)$.

CAUTION!: $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$ is a closed simple curve. but
$q(t) = (\cos t, \sin t)$; $0 \le t \le 4\pi$ is NOT a simple curve, hence $\int_c F\cdot ds \neq \int_q F\cdot ds$.

Notation: Let C be a curve. then $-C$ or C^- is a curve with opposite orientation to
Moreover $\int_C F\cdot ds = -\int_{C^-} F\cdot ds$.

Notation: $C = $ "$C_1 + C_2 + \dots + C_k$", e.g  , then $\int_C F\cdot ds = \int_{C_1} F\cdot ds + \int_{C_2} F\cdot ds + \cdots + \int_{C_n} F\cdot ds$

arametrization of Surfaces

parametrization of the surface is a mapping $\Phi: D \to \mathbb{R}^3$, where $D \subset \mathbb{R}^2$.
$S = \Phi(D)$ is the corresponding surface.

If Φ is C^1 then S is called C^1-smooth and Φ is regular.

s a matter of notation $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$.

ef: We say that S (S is the image of the parametrization of Φ: $S = \Phi(D)$)
regular or smooth at $\Phi(u_0,v_0)$ if T_u and T_v are independent, i.e., $T_u \times T_v \neq 0$.
is regular if it is regular at every point.

angent Plane: If a parametrized surface $S = \Phi(D)$ is regular at $\Phi(u_0,v_0)$,
hen the tangent plane to S at $\Phi(u_0,v_0)$ is the plane spanned by T_u and T_v.

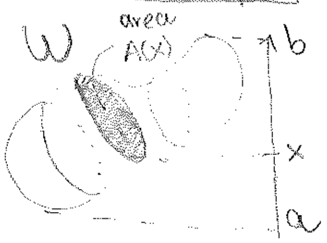
$\vec{n} = T_u \times T_v(u_0,v_0)$; $\vec{n} \cdot (x-u_0, y-v_0, z-\Phi(u_0,v_0)) = 0$. ⊢ EXAM 1 ⊣

EVIEW CHAPTER 5: Double-Triple Integrals.

$f: D \subset \mathbb{R}^2 \to \mathbb{R}^1$. Then $\iint_D f(x,y)\,dA$ is the volume under the surface f.

$\iint_D dA = \text{Area}(D)$. [some results for triple integrals].

AVALIERI Principle:



$W \subset \mathbb{R}^3$.

$\text{Vol}(W) = \int_a^b A(x)\,dx$. (or if you can slice w.r.t. y-axis: $\text{Vol}(W) = \int_c^d A(y)\,dy$).

ini's THEOREM: $f: [a,b] \times [c,d] \to \mathbb{R}$; f-continuous:

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx = \iint_R f(x,y)\,dA.$$

APTER 6: CHANGE OF VARIABLES AND Applications of integration.

$f: A \to B$ be a function.
is one-to-one if given $x, y \in A$ $f(x) = f(y) \Rightarrow x = y$
is onto if for every $b \in B$ there exists $a \in A$ s.t. $f(a) = b$
change of variables we will be mostly concerned with bijective (1-1, onto) maps.
change of variables is linear.

ear mappings $\mathbb{R}^2 \to \mathbb{R}^2$: $T: \mathbb{R}^2 \to \mathbb{R}^2$; given by matrix multiplication
$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.

EOREM: $A \in M_2(\mathbb{R})$; $\det(A) \neq 0$; T linear s.t. $T(x) = Ax$. then T transforms
allelograms into parallelograms and vertices into vertices. Moreover, if $T(D^*)$
a parallelogram then D^* must be a parallelogram.

Definition: Jacobian Determinant: Let $T: D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 transformation given by $x = x(u,v)$ and $y = y(u,v)$. The jacobian determinant of T written $\frac{\partial(x,y)}{\partial(u,v)}$ is the determinant of the derivative matrix $DT(u,v)$ of T

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

CHANGE OF VARIABLES FORMULA: $D, D^* \subset \mathbb{R}^2$; $T: D^* \to D$; C^1, 1-1 and onto then

$f: D \to \mathbb{R}$;

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} \underbrace{f(x(u,v), y(u,v))}_{f(T(u,v))} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

Similarly for $\mathbb{R}^3$:

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(T(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

NEMONIC: $dx\,dy \to du\,dv$ then $dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$ "canceling $\partial(u,v)$ with $du\,dv$"

three uses of change of variables formula:

(i) Polar coordinates. $x = r\cos\theta$; $y = r\sin\theta$ $\Rightarrow dx\,dy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = r\,dr\,d\theta$.

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

(ii) Cylindrical coordinates:

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz.$$

(iii) Spherical coordinates:

$x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi.$$

Applications:

(I) Average value of f over a region

(II) Center of mass of a solid

(III) Moments of inertia of a solid

(IV) Gravitational potential of a solid.

I) Average Value:

$$\frac{\iiint_R f\,dV}{\iiint_R dV} \quad \text{and} \quad \frac{\iint_R f\,dA}{\iint_R dA}$$

Vol of R → $\iiint_R dV$; Area of R → $\iint_R dA$

$f : [a,b] \to \mathbb{R}$, $[f]_{av} = \frac{1}{b-a}\int_a^b f(x)dx$

I) Center of mass: $(\bar{x}, \bar{y}, \bar{z})$ (same for two dimensions. $\iiint_R \delta(x,y,z)dV = \text{mass}$)

$$\bar{x} = \frac{\iiint_R x\,\delta(x,y,z)dV}{\iiint_R \delta(x,y,z)dV} \; ; \; \bar{y} = \frac{\iiint_R y\,\delta(x,y,z)dV}{\iiint_R \delta(x,y,z)dV} \; ; \; \bar{z} = \frac{\iiint_R z\,\delta(x,y,z)dV}{\iiint_R \delta(x,y,z)dV}.$$

I) Moments of inertia:

$$I_x = \iiint_W (y^2+z^2)\delta(x,y,z)dV \; ; \; I_y = \iiint_W (x^2+z^2)\delta(x,y,z)dV \; ; \; I_z = \iiint_W (x^2+y^2)\delta(x,y,z)dV$$

MNEMONIC: missing variable.

I) Gravitational Potential $V = -GmM\left[\frac{1}{r}\right]_{av}$; $\left[\frac{1}{r}\right]_{av} = \frac{1}{vol(W)}\iiint_W \frac{1}{r}dV.$

BACK TO CHAPTER 7.

7.4 Computing areas of surfaces in $\mathbb{R}^3$: (Area is independent of parametrization)

Given a surface $S \subset \mathbb{R}^3$; we first need a parametrization $S = \Phi(D)$, over some region $D \subset \mathbb{R}^2$. So that $\Phi : D^2 \subset \mathbb{R}^2 \to \mathbb{R}^3$. the parametrization must satisfy

- Φ is one-to-one.
- Φ is regular, i.e., Φ is C^1 and $T_u \times T_v \neq 0$; $T_u = \frac{\partial\Phi}{\partial u}$; $T_v = \frac{\partial\Phi}{\partial v}$

Notation: $\Phi(u,v) = (x(u,v), y(u,v), z(u,v)) \iff \Phi = (x,y,z)$.

Definition: the area of the surface $S = \Phi(D)$ (under above assumptions) is given by:

$$\boxed{A(S) = \iint_D \|T_u \times T_v\| du\,dv}, \quad \text{where}$$

$$\|T_u \times T_v\| = \sqrt{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2}$$

Remember: $\Phi' = \begin{pmatrix} \frac{\partial\Phi}{\partial u} \\ \frac{\partial\Phi}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix}$

Surface area of a Graph: $z = g(x,y)$. $(x,y) \in D$:

$$\boxed{A(S) = \iint_D \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\, dA}$$

Follows from above by the parametrization $x = u$; $y = v$; $z = g(u,v)$.

Surfaces of revolution.

about the x-axis:

$$A = 2\pi \int_a^b \left(|f(x)|\sqrt{1+[f'(x)]^2}\right)dx.$$

about the y-axis:

$$A = 2\pi \int_a^b \left(|x|\sqrt{1+[f'(x)]^2}\right)dx.$$

Integrals of Scalar functions over surfaces:

$\Phi : D \to S.$

(I) If $f(x,y,z) : S \to \mathbb{R}$; S a surface, integral of f over S to be:

$$\iint_S f\, dS = \iint_D f(\Phi(u,v))\, \|T_u \times T_v\|\, du\, dv$$

Nemonic: (i) path integral
(ii) if $f \equiv 1$ then we get the area of surface S.

(II) If $z = g(x,y)$ $g: D \subset \mathbb{R}^2 \to \mathbb{R}$. then parametrize:
$\Phi(u,v) = (u, v, g(u,v))$; $(u,v) \in D$. then apply previous formula to get

$$\iint_S f\, dS = \iint_D f(u,v,g(u,v)) \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2}\, du\, dv$$

(III) Again, If $z = g(x,y)$ then

$$\iint_S f\, dS = \iint_D f(u,v,g(u,v)) \frac{du\, dv}{\hat{n} \cdot k}$$

, where $\hat{n} = \frac{N}{\|N\|}$;
$N = \nabla(z - g(x,y)) = \langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \rangle$

Integrals of VECTOR FIELDS over surfaces: (surface integrals of VECTOR Fields)

Let S be a surface in $\mathbb{R}^3$. ($S \subset \mathbb{R}^3$) Let F - vector field on S. Let Φ be a parametrization of S. $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$. $S = \Phi(D)$.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\Phi(u,v)) \cdot (T_u \times T_v)\, du\, dv$$

(depends only on $\vec{F}$, S and the orientation of S)

Def: A regular surface S is called <u>orientable</u> if there exists a <u>continuous</u> vector field $\vec{n}$ on S such that $\vec{n}(p)$ is a unit normal vector to S at p. This vector field is called an orientation.

We say that Φ is <u>orientation preserving</u> if

$$\frac{T_u \times T_v}{\|T_u \times T_v\|} = \vec{n}(\Phi(u,v)) \text{ ; for all } (u,v) \in D$$

We say that Φ is <u>orientation reversing</u> if

$$\frac{T_u \times T_v}{\|T_u \times T_v\|} = -\vec{n}(\Phi(u,v)) \text{ ; for all } (u,v) \in D$$

THEOREM: If Φ_1, Φ_2 are regular orientation preserving parametrizations of S,

$$\iint_{\Phi_1} \vec{F} \cdot d\vec{S} = \iint_{\Phi_2} \vec{F} \cdot d\vec{S}$$

If Φ_1 is orientation preserving and Φ_2 is orientation reversing

$$\iint_{\Phi_1} \vec{F} \cdot d\vec{S} = -\iint_{\Phi_2} \vec{F} \cdot d\vec{S}$$

lowever: $\iint_{\Phi_1} f\,dS = \iint_{\Phi_2} f\,dS$ regardless of orientations.

THEOREM:

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS$$

, where $\vec{n}$ is the normal vector to the surface. So that $\vec{F}\cdot\vec{n}$ is normal component of $\vec{F}$

CURVATURE:

(I) Gauss curvature

$$K(p) = \frac{ln - m^2}{EG - F^2}$$

(II) mean curvature

$$H(p) = \frac{Gl + En - 2Fm}{2(EG - F^2)}$$

where; $E = \|\Phi_u\|^2$; $F = \|\Phi_{uv}\|^2$; $G = \|\Phi_v\|^2$

$l = N\cdot\Phi_{uu}$; $m = N\cdot\Phi_{uv}$, $n = N\cdot\Phi_{vv}$

P_1 = minimal "directional" curvature
P_2 = maximal "directional" curvature

$K = P_1 P_2$

$H = \frac{P_1 + P_2}{2}$

Ways to remember:
First Fundamental Form: $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = I$
Second Fundamental Form: $\begin{pmatrix} l & m \\ m & n \end{pmatrix} = II$

GAUSS curvature: $K(p) = \frac{\det II}{\det I} = \frac{ln - m^2}{EG - F^2}$

THEOREM (Gauss-Bonnet): If S has genus g (# of holes) then

$$\frac{1}{2\pi}\iint_S K\,dA = 2 - 2g$$

CHAPTER 8: the "integral theorems of vector Analysis"

Green's theorem: Let D be a simple region and let C be its boundary, suppose $P: D \to \mathbb{R}$ and $Q: D \to \mathbb{R}$ are of class C^1. then:

$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Note: C^+ means the region is on your left.

In general we can apply Green's theorem to any reasonable region by dividing it appropriately.

Application: Area of a region enclosed by $C = \partial D$.

$$A = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx$$

theorem: Vector Form of Green's theorem. Using Green's theorem and the fact that $\left\{ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \text{scalar curl of } \vec{F} = \text{curl}\,\vec{F} \cdot \hat{k} = \nabla \times \vec{F} \cdot \hat{k} \right.$

$$\int_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D (\text{curl}\,\vec{F}) \cdot \hat{k}\, dA = \iint_D (\nabla \times \vec{F}) \cdot \hat{k}\, dA$$

Divergence theorem (In the plane): Let $D \subset \mathbb{R}^2$ be a region to which Green's theorem applies and let ∂D be its boundary. Let $\vec{n}$ denote the outward unit normal to ∂D. Then:

$$\int_{\partial D} \vec{F} \cdot \vec{n} = \iint_D \text{div}\,\vec{F}\, dA$$

where: $\vec{F} = P\hat{i} + Q\hat{j}$

$$\vec{n} = \frac{\langle y'(t), -x'(t) \rangle}{\| \langle x'(t), y'(t) \rangle \|}$$

Stoke's theorem: Parametrized surfaces.

Let S be an oriented surface defined by a one-to-one parametrization $\Phi: D \subset \mathbb{R}^2 \to S$, where D is a region to which Green's thm applies. Let ∂S denote the oriented boundary of S and let $\vec{F}$ be a C^1 vector field on S. Then

$$\iint_S (\nabla \times \vec{F})\, d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$$

FACT: If S is a closed surface that does not have a boundary, then the integral is equal to zero.

EX: Sphere or ellipsoid, then the integral is zero

(Analogous to: the line integral of a gradient field over a closed curve is zero.)

In this case $\vec{F}$ must be the curl of some other vector field.

CONSERVATIVE FIELDS:

theorem: Let $\vec{F}$ be a C^1 v.f. on $\mathbb{R}^3$ except for possibly finitely many points.

then TFCAE

i) $\int_C \vec{F} \cdot d\vec{s} = 0$, for any oriented simple closed curve C.

ii) $\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}$, for any simple oriented curves C_1, C_2 with the same end points

iii) $\vec{F}$ is a gradient vector field.

iv) $\text{curl}\,\vec{F} = \vec{0}$

A vector field satisfying one (and, hence, all) of the conditions (i)-(iv) is called a __conservative vector field__.

In the planar case: $\vec{F} = (F_1, F_2)$ v.f on $\mathbb{R}^2$, C^1. TFCAE.

i) $\int_C \vec{F} d\vec{s} = 0$, C-closed.

ii) path independent.

iii) $\vec{F} = \nabla f$

iv) $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ (scalar curl $= 0$).

{Need to be C^1 on ALL of $\mathbb{R}^2$, otherwise the theorem does not apply. Unlike previous thm on $\mathbb{R}^3$ where we allow a finite number of exceptional points}

GAUSS' THEOREM (Divergence Theorem)

LET W be a symmetric elementary region in space. Denote by ∂W the oriented closed surface that bounds W. Let $\vec{F}$ be a smooth v.f. defined on W. Then

$$\iiint_W \operatorname{div} \vec{F} \, dV = \iint_{\partial W} \vec{F} \cdot d\vec{S} = \iint_{\partial W} (\vec{F} \cdot \vec{n}) \, d\vec{S}$$

GAUSS' LAW: Let W be a symmetric elementary region in $\mathbb{R}^3$. If $(0,0,0) \notin \partial W$

$$\iint_{\partial W} \frac{\vec{r} \cdot \vec{n}}{r^3} \, dS = \begin{cases} 4\pi & \text{if } (0,0,0) \in W \\ 0 & \text{if } (0,0,0) \notin W. \end{cases}$$

Differential Forms:

0-forms: functions. $f = f(x,y,z)$.

1-forms: $P\,dx + Q\,dy + R\,dz$, where $P = P(x,y,z)$, $Q = Q(x,y,z)$, $R = R(x,y,z)$.

2-forms: $F\,dxdy + G\,dxdz + H\,dzdy$

3-forms: $F\,dxdydz$

integrate k-forms over k-dimension

Algebra of forms: PAGE 483.

EX: $\int_C (x+y)dx + (2x-z)dy + (y+z)dz$

$= \iint_S d((x+y)dx + (2x-z)dy + (y+z)dz)$

$= \iint_S dxdy + 2dydz$