SEction 15.3:

(2) $\displaystyle\int_0^1\int_{2x}^2 (x-y)\,dy\,dx = \int_0^1 \left(xy - \frac{y^2}{2}\right)_{2x}^2 dx$

$\displaystyle= \int_0^1 (2x - 2 - \cancel{2x^2} + 2x^2)\,dx = \int_0^1 2x-2\,dx = (x^2-2x)_0^1$

$= 1-2 = -1$

(8) $\displaystyle\iint_D \frac{y}{x^5+1}\,dA$, $D=\{(x,y) \mid 0\le x\le 1 , 0\le y\le x^2\}$

$\displaystyle\int_0^1\int_0^{x^2} \frac{y}{x^5+1}\,dy\,dx = \int_0^1 \frac{1}{x^5+1}\left[\int_0^{x^2} y\,dy\right]dx = \int_0^1 \frac{1}{x^5+1}\left(\frac{y^2}{2}\right)_0^{x^2} dx$

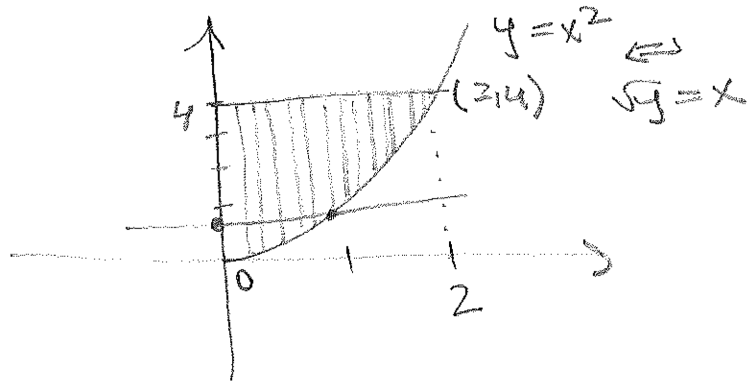
$\displaystyle=\frac{1}{2}\int_0^1 \frac{x^4}{x^5+1}\,dx$ substitute: $u = x^5+1 \Rightarrow du = 5x^4\,dx \Rightarrow x^4\,dx = \frac{du}{5}$

$\displaystyle\frac{1}{2}\int_0^{\square} \frac{1\cdot du}{u\cdot 5} = \frac{1}{10}\int_0^{\square}\frac{du}{u} = \frac{1}{10}\left[\ln(u)\right]_{\square}^{\square} = \frac{1}{10}\left[\ln(x^5+1)\right]_0^1$

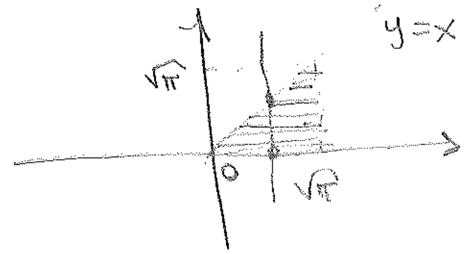
$\displaystyle= \frac{1}{10}\left(\ln(2) - \ln(1)\right) = \frac{1}{10}\left(\ln(2) - 0\right) = \boxed{\frac{\ln(2)}{10}}$

(44) $\displaystyle\int_0^2\int_{x^2}^4 f(x,y)\,dy\,dx$

$\displaystyle= \int_0^4\int_0^{\sqrt{y}} f(x,y)\,dx\,dy$



(50) $\displaystyle\int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}} \cos(x^2)\,dx\,dy$



$$= \int_0^{\sqrt{\pi}}\int_0^{x} \cos(x^2)\,dy\,dx = \int_0^{\sqrt{\pi}} \cos(x^2)\left[\int_0^{x} dy\right]dx$$

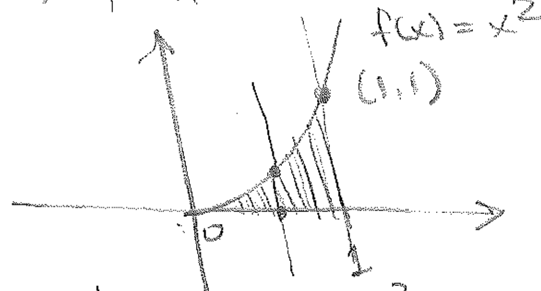
$$= \int_0^{\sqrt{\pi}} \cos(x^2)\,[y]_0^{x}\,dx = \int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$$

Substitute: $u = x^2 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$

$$\rightarrow \int_0^0 \cos(u)\frac{du}{2} = \frac{1}{2}\int_0^0 \cos(u)\,du = \frac{1}{2}[\sin(u)]_0^0$$

$$\rightarrow \frac{1}{2}[\sin(x^2)]_0^{\sqrt{\pi}} = \frac{1}{2}(\sin(\pi)-\sin(0)) = \boxed{0}$$

(60) $f(x,y) = x\sin y$, D is enclosed by the curves $y=0$, $y=x^2$, and $x=1$



the average value Ave, is given by:

$$\text{Ave} = \frac{\text{Vol}}{\text{Area}}, \text{ Hence}$$

$$\text{Ave} = \frac{\int_0^1\int_0^{x^2} x\sin y\,dy\,dx}{\int_0^1 x^2\,dx} = \frac{\int_0^1 x\left[\int_0^{x^2}\sin y\,dy\right]dx}{\left[\frac{x^3}{3}\right]_0^1} = \frac{\int_0^1 x\,[-\cos(y)]_0^{x^2}\,dx}{\frac{1}{3}}$$

$$= 3\int_0^1 x(-\cos(x^2)+\cos(0))\,dx = 3\int_0^1 x - x\cos(x^2)\,dx$$

$$= 3\Big[\underbrace{\int_0^1 x\,dx}_{A} - \underbrace{\int_0^1 x\cos(x^2)\,dx}_{B}\Big] : \quad A = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$B = \int_0^1 x\cos(x^2)\,dx$ Substitute: $x^2 = u \Rightarrow 2x\,dx = du \Rightarrow \frac{du}{2} = x\,dx$

$$\rightarrow \int_0^0 \cos(u)\frac{du}{2} = \frac{1}{2}\int_0^0 \cos(u)\,du = \frac{1}{2}[\sin(u)]_0^0 \rightarrow \frac{1}{2}[\sin(x^2)]_0^1 = \frac{1}{2}(\sin(1))$$

Hence, $3(A-B) = 3\left[\left(\frac{1}{2}\right) - \frac{1}{2}\sin(1)\right] = 3\left(\frac{1-\sin(1)}{2}\right) = \boxed{\frac{3}{2}(1-\sin(1))}$

SECTION 15.4:

(12) $\iint_D \cos(\sqrt{x^2+y^2})\,dA$, $D = \{(x,y) \mid x^2+y^2 \leq 2^2\}$

Changing to polar coordinates:

$$\int_0^{2\pi}\int_0^2 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \cos(\sqrt{r^2})\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r\cos(r)\,dr\,d\theta = \int_0^{2\pi} \Big[\cos(r) + r\sin(r)\Big]_0^2 d\theta$$

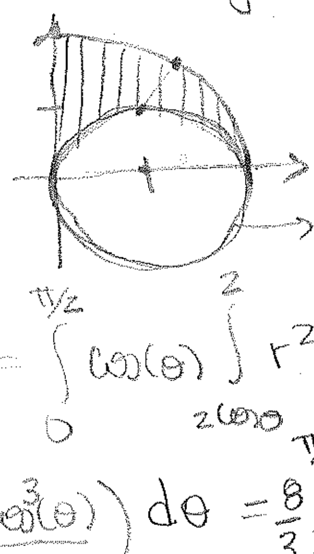
$$= \int_0^{2\pi} \big(\cos(2) + 2\sin(2)\big) - \big(\overset{1}{\cos(0)} + \cancel{0\cdot\sin(0)}\big)\,d\theta$$

$$= \int_0^{2\pi} \cos(2) + 2\sin(2) - 1\,d\theta = \boxed{\big(\cos(2) + 2\sin(2) - 1\big)\cdot 2\pi}$$

(14) $\iint_D x\,dA$, D: is the region in the first quadrant that lies between the circles $x^2+y^2=4$ and $x^2+y^2=2x \iff x^2-2x+y^2=0$

$\iff (x-1)^2+y^2=1$ circle centered at $(1,0)$, radius 1.

$$= \int_0^{\pi/2}\int_{2\cos\theta}^{2} (r\cos(\theta))\,r\,dr\,d\theta$$



$$= \int_0^{\pi/2}\int_{2\cos\theta}^{2} r^2\cos(\theta)\,dr\,d\theta = \int_0^{\pi/2}\cos(\theta)\int_{2\cos\theta}^{2} r^2\,dr\,d\theta = \int_0^{\pi/2}\cos\theta\left[\frac{r^3}{3}\right]_{2\cos\theta}^{2} d\theta$$

$$= \int_0^{\pi/2}\cos\theta\left(\frac{8}{3} - \frac{8\cos^3(\theta)}{3}\right)d\theta = \frac{8}{3}\int_0^{\pi/2}\cos(\theta) - \cos^3(\theta)\,d\theta$$

$$= \frac{8}{3}\left[\sin(\theta) - \left(\frac{1}{3}(2+\cos^2(\theta))\sin(\theta)\right)\right]_0^{\pi/2}$$

$$= \frac{8}{3}\left(\left(1 - \left(\frac{1}{3}(2)\right)\right) - (0-0)\right) = \frac{8}{3}\left(1-\frac{2}{3}\right) = \frac{8}{3}\cdot\frac{1}{3} = \boxed{\frac{8}{9}}$$

(20) Below the paraboloid $z = 18 - 2x^2 - 2y^2$
and above the xy-plane

First, find the domain of integration:

$$\text{xy-plane} \iff z = 0$$

$$0 = 18 - 2x^2 - 2y^2 \iff 0 = 2(9 - x^2 - y^2)$$

$$\iff x^2 + y^2 = 3^2$$

circle centered at the origin with radius 3.

$$\iint_D f(x,y)\,dA = \int_0^{2\pi}\int_0^3 f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

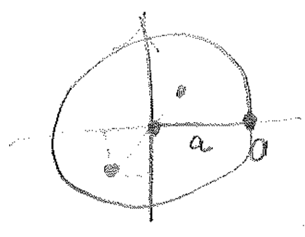
$$= \int_0^{2\pi}\int_0^3 (18 - 2r^2\cos^2\theta - 2r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (18 - 2r^2(\underbrace{\cos^2\theta + \sin^2\theta}_{1}))\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 18r - 2r^3\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(9r^2 - \frac{r^4}{2}\right)\Big|_0^3 d\theta = \int_0^{2\pi}\frac{81}{2}\,d\theta = \boxed{81\pi}$$

(38)



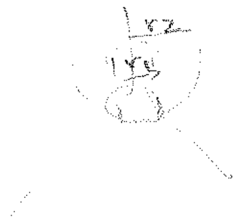
Let (r,θ) denote the distance of an arbitrary point inside the circle (i.e., $0 \le r \le a$).
$0 \le \theta \le 2\pi$

then the distance of this point to the origin is r.

$$\text{Average} = \frac{\text{Vol}}{\text{Area}} = \frac{\iint r\,dA}{\pi a^2} = \frac{\int_0^{2\pi}\int_0^a r^2\,dr\,d\theta}{\pi a^2} = \frac{\int_0^{2\pi}\left[\frac{r^3}{3}\right]_0^a d\theta}{\pi a^2}$$

$$= \frac{\int_0^{2\pi}\frac{a^3}{3}\,d\theta}{\pi a^2} = \frac{\frac{a^3}{3}\cdot 2\pi}{\pi \cdot a^2} = \frac{\frac{2\pi a^3}{3}}{a^2\cdot\pi} = \frac{2\cancel{\pi} a^3}{3\cancel{\pi} a^2} = \boxed{\frac{2}{3}a}$$

(28)



Let the sphere with radius

volume of sphere $= \dfrac{4\pi r^3}{3} = \boxed{\dfrac{4\pi r_2^3}{3}}$

volume of cylinder $= \pi \cdot r^2 \cdot L = \boxed{\pi \cdot r_1^2 \cdot h}$

$V(\text{ring}) = \frac{4}{3}\pi r_2^3 - h \cdot \pi \cdot r_1^2 - 2\pi \cdot h(3r_1^2 + h^2)$

$V\ \text{cap} = \pi \cdot h(3r_1^2 + h^2)$

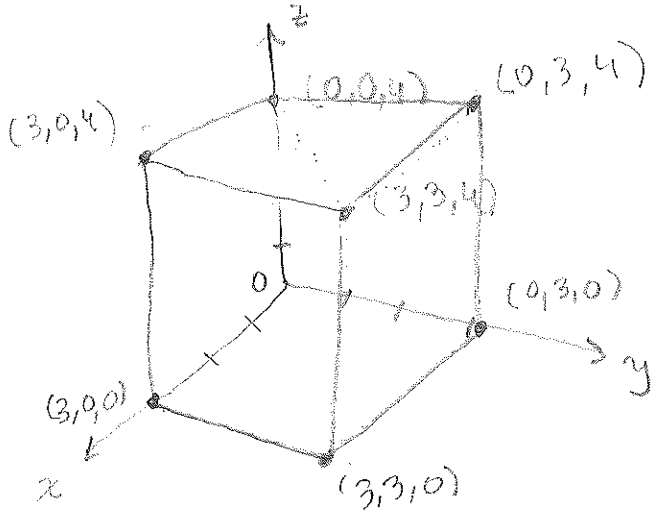
$\boxed{= \frac{4}{3}\pi r_2^3 - h\pi r_1^2 - 6\pi h r_1^2 - 2\pi h^3}$

(Similar to 4) ✱
Exercise Similar to 12.1 (4)

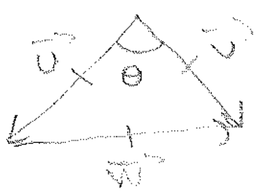
$P(3,3,4)$

the projections of P are:

On the xy-plane $(3,3,0)$
On the yz-plane $(0,3,4)$
on the xz-plane $(3,0,4)$



✱ (LAST EXERCISE IF TIME PERMITS)
Exercise 11 Section 12.3



$|\vec{u}| = 1 \Rightarrow \theta = 60°$

$\Rightarrow \vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta = 1\cdot 1\cdot\cos 60° = \frac{1}{2}$

(Similar to 26)
Find a vector that has the same direction as $\vec{w} = \langle -1,2,3\rangle$ but has length 4.

Let $\vec{v}$ be a vector with the same direction as $\vec{w}$ but with length 4. then

$$\vec{v} = 4\cdot\frac{\vec{w}}{|\vec{w}|} = \frac{4\ \langle -1,2,3\rangle}{\sqrt{(-1)^2+2^2+3^2}} = \frac{4\ \langle -1,2,3\rangle}{\sqrt{1+4+9}} = \frac{4}{\sqrt{14}}\cdot\langle -1,2,3\rangle$$

We can check that indeed $|\vec{v}| = \sqrt{\left(\frac{-4}{\sqrt{14}}\right)^2+\left(\frac{8}{\sqrt{14}}\right)^2+\left(\frac{12}{\sqrt{14}}\right)^2} = \sqrt{\frac{16}{14}+\frac{64}{14}+\frac{144}{14}}$
$= \sqrt{\frac{224}{14}} = \sqrt{16} = 4$ ✓

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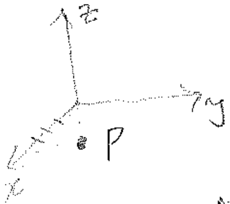
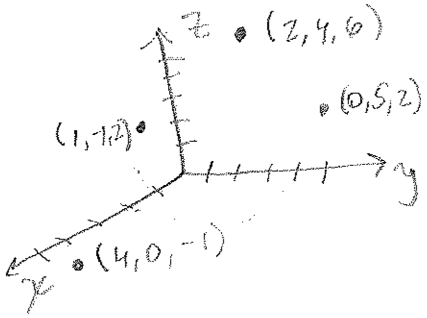
SECTION 12.1:

(1) $P(4,0,-3)$.

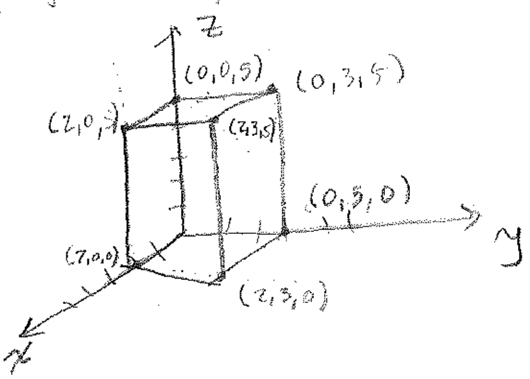


(2)



(3) the point $A(-4,0,-1)$ lies on the xz-plane.
the point $C(2,4,6)$ is the closest to the yz-plane

(4) Let $P(2,3,5)$. the projection of P on the xy-plane is $(2,3,0)$
the projection of P on the yz-plane is $(0,3,5)$.
the projection of P on the xz-plane is $(2,0,5)$.

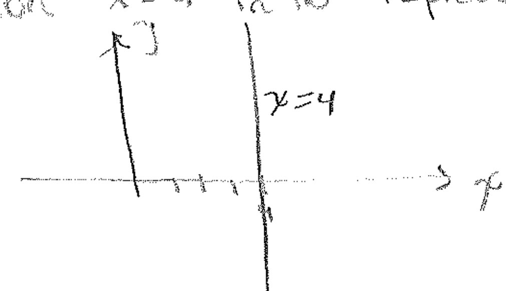


the length of the diagonal of the box is

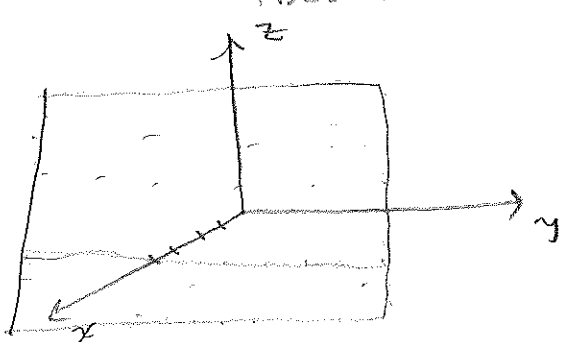
$$d(P,O) = \sqrt{(2-0)^2+(3-0)^2+(5-0)^2}$$
$$= \sqrt{2^2+3^2+5^2}$$
$$= \sqrt{4+9+25} = \sqrt{13+25} = \sqrt{38}$$

(5) the graph is a plane

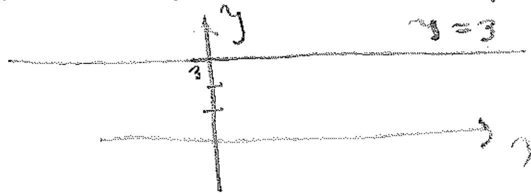
(6) (a) the equation $x=4$ in $\mathbb{R}^2$ represents a vertical line as follow:



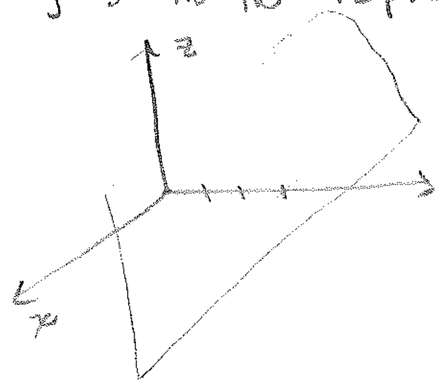
the same equation $x=4$, but in $\mathbb{R}^3$ represents a plane as follow.



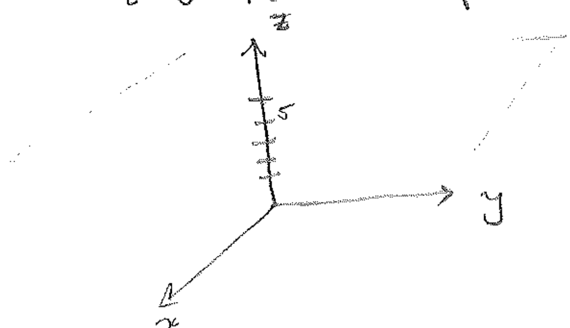
(b) the equation $y=3$ in $\mathbb{R}^2$ represents a horizontal line as follow:



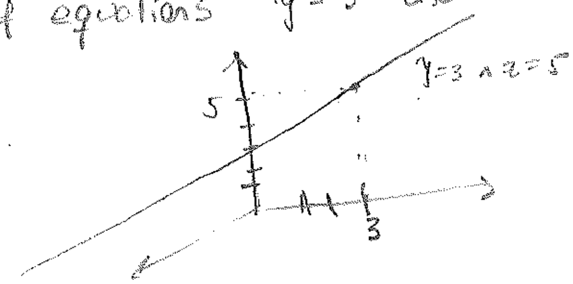
the equation $y=3$ in $\mathbb{R}^3$ represents a plane as follow:



the equation $z=5$ in $\mathbb{R}^3$ represents a horizontal plane as follow:



the pair of equations $y=3$ and $z=5$, in $\mathbb{R}^3$, represent a line as follow:



(7) Find the lengths of the sides of the triangle PQR, where $P(3,-2,-3)$, $Q(7,0,1)$, $R(1,2,1)$.

First construct the vectors $\vec{PQ} = Q-P = (7,0,1)-(3,-2,-3) = (4,2,4)$

and $\vec{PR} = R-P = (1,2,1)-(3,-2,-3) = (-2,4,4)$

and $\vec{RQ} = Q-R = (7,0,1)-(1,2,1) = (6,-2,0)$

the respectives lengths are: $|\vec{PQ}| = \sqrt{4^2+2^2+4^2} = \sqrt{36} = 6$

$|\vec{PR}| = \sqrt{2^2+4^2+4^2} = \sqrt{36} = 6$

$|\vec{RQ}| = \sqrt{6^2+2^2+0^2} = \sqrt{40} = 2\sqrt{10}$

Hence, the triangle PQR is an isosceles triangle.

It is not a right triangle since it does not satisfy the Pythagorean theorem.

(8) $\vec{PQ} = Q - P = (4,1,1) - (2,-1,0) = (2,2,1)$

$\vec{PR} = R - P = (4,-5,4) - (2,-1,0) = (2,-4,4)$

$\vec{RQ} = Q - R = (4,1,1) - (4,-5,4) = (0,6,-3)$

Then, $|\vec{PQ}| = \sqrt{2^2+2^2+1^2} = \sqrt{9} = 3$

$|\vec{PR}| = \sqrt{2^2+4^2+4^2} = \sqrt{36} = 6$

$|\vec{RQ}| = \sqrt{0+6^2+3^2} = \sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5}$

this is a right-triangle since $|\vec{PQ}|^2 + |\vec{PR}|^2 = |\vec{RQ}|^2$

It is not isosceles $3^2 + 6^2 = (\sqrt{45})^2$

(9)

(a) $\begin{vmatrix} 2 & 4 & 2 \\ 3 & 7 & -2 \\ 1 & 3 & 3 \end{vmatrix} = 2(21+6) - 4(9+2) + 2(9-7) = 2(27) - 4(11) + 2(2)$

$= 54 - 44 + 4 = 10 + 4 = 14 \neq 0$

these are not in a straight line.

(b) $\begin{vmatrix} 0 & -5 & 5 \\ 1 & -2 & 4 \\ 3 & 4 & 2 \end{vmatrix} = 0(\ ) + 5(2-12) + 5(4+6) = 5(-10) + 5(10) = 0$

these are colinear.

(10) Find the distance from $(4,-2,6)$ to each of the following:

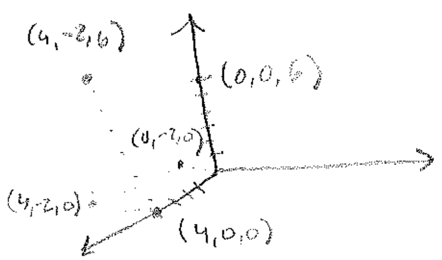
(a) the xy-plane $= 6$

(b) the yz-plane $= 4$

(c) the xz-plane $= 2$

(d) the x-axis $= d((4,-2,6),(4,0,0))$

$= \sqrt{2^2+6^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$



(e) the y-axis $= d((4,-2,6),(0,-2,0))$

$= \sqrt{4^2+6^2} = \sqrt{16+36} = \sqrt{52} = 2\sqrt{13}$

(f) the z-axis $= d((4,-2,6),(0,0,6))$

$= \sqrt{4^2+2^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$

(11) $(x+3)^2 + (y-2)^2 + (z-5)^2 = 4^2$

The yz-plane are all the points $\{(0,y,z) : y,z \in \mathbb{R}\}$

Hence, the intersection of this sphere with the yz-plane is the circle

$(y-2)^2 + (z-5)^2 = 4^2 - 3^2 = 16 - 9 = 7.$

the intersection of this sphere with the xy-plane is:

$(x+3)^2 + (y-2)^2 = 4^2 - 5^2 < 0 \Rightarrow$ there is no intersection

the intersection of this sphere with the xz-plane is the circle

$(x+3)^2 + (z-5)^2 = 4^2 - 2^2 = 16 - 4 = 12.$

(12) $(x-2)^2+(y+6)^2+(z-4)^2=5^2$

Intersections: with xy-plane: $(x-2)^2+(y+6)^2=5^2-4^2=25-16=9=3^2$

with xz-plane: $(x-2)^2+(z-4)^2=5^2-6^2<0 \Rightarrow$ No intersection

with yz-plane: $(y+6)^2+(z-4)^2=5^2-2^2=25-4=21$

(13) $d((3,8,1),(4,3,-1))=\sqrt{1^2+5^2+2^2}=\sqrt{30}=r$

Hence, the equation of the sphere that passes through the point $(4,3,-1)$ and has center $(3,8,1)$ is: $(x-3)^2+(y-8)^2+(z-1)^2=(\sqrt{30})^2=30$

(14) $d(0,(1,2,3))=\sqrt{1^2+2^2+3^2}=\sqrt{1+4+9}=\sqrt{14}$

Hence, the equation of the sphere that passes through the origin and has center $(1,2,3)$ is: $(x-1)^2+(y-2)^2+(z-3)^2=(\sqrt{14})^2=14$

(15) $x^2+y^2+z^2-2x-4y+8z=15$

$= x^2-2x+y^2-4y+z^2+8z$

$=(x^2-2x+1)-1+(y^2-4y+4)-4+(z^2+8z+16)-16$

$=(x-1)^2-1+(y-2)^2-4+(z+4)^2-16$

$=(x-1)^2+(y-2)^2+(z+4)^2-21$

$\Rightarrow (x-1)^2+(y-2)^2+(z+4)^2=15+21=36$

the sphere's center is $C(1,2,-4)$ and the radius is $\sqrt{36}=6$

(16) $x^2+y^2+z^2+8x-6y+2z+17=0$

$=x^2+8x+y^2-6y+z^2+2z+17$

$=(x+4)^2-16+(y-3)^2-9+(z+1)^2-1+17$

$\Rightarrow (x+4)^2+(y-3)^2+(z+1)^2=9$

the sphere's center is $C(-4,3,1)$ and the radius is $\sqrt{9}=3$

(19) (a) Let $M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$.

Since, $d(M,P_1)=\sqrt{\left(\frac{x_1+x_2}{2}-x_1\right)^2+\left(\frac{y_1+y_2}{2}-y_1\right)^2+\left(\frac{z_1+z_2}{2}-z_1\right)^2}$

$=\sqrt{\left(\frac{x_1+x_2-2x_1}{2}\right)^2+\left(\frac{y_1+y_2-2y_1}{2}\right)^2+\left(\frac{z_1+z_2-2z_1}{2}\right)^2}$

$=\sqrt{\left(\frac{x_2-x_1}{2}\right)^2+\left(\frac{y_2-y_1}{2}\right)^2+\left(\frac{z_2-z_1}{2}\right)^2}=\sqrt{\frac{(x_2-x_1)^2}{4}+\frac{(y_2-y_1)^2}{4}+\frac{(z_2-z_1)^2}{4}}$

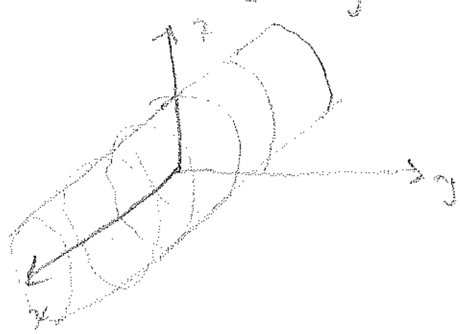
Now, since $(x_2-x_1)^2=(x_1-x_2)^2$ we have that $=\sqrt{\frac{(x_1-x_2)^2}{4}+\frac{(y_1-y_2)^2}{4}+\frac{(z_1-z_2)^2}{4}}$

$=\sqrt{\left(\frac{x_1-x_2}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2+\left(\frac{z_1-z_2}{2}\right)^2}=\sqrt{\left(\frac{x_1+x_2}{2}-x_2\right)^2+\left(\frac{y_1+y_2}{2}-y_2\right)^2+\left(\frac{z_1+z_2}{2}-z_2\right)^2}$

$=d(M,P_2) \Leftrightarrow d(M,P_1)=d(M,P_2) \Rightarrow M$ is the midpoint ...

(30): Describe in words the region of $\mathbb{R}^3$ represented by the equation: ③

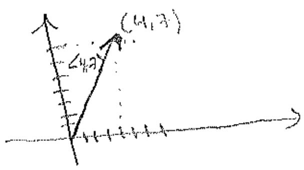
$y^2+z^2=16$. Given an x, the equation represents a circle of radius 4. Put all circles toghether to get a cylinder like



Section 12.2:

(1) (a) scalar, (b) vector, (c) vector, (d) scalar

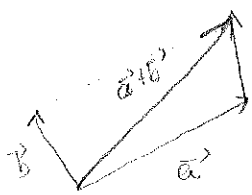
(2)



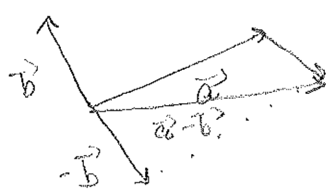
(3) $\vec{DA}=\vec{CB}$, $\vec{DC}=\vec{AB}$, $\vec{EA}=\vec{CE}$, $\vec{DE}=\vec{EB}$

(6)

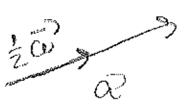
(a) $\vec{a}+\vec{b}=$



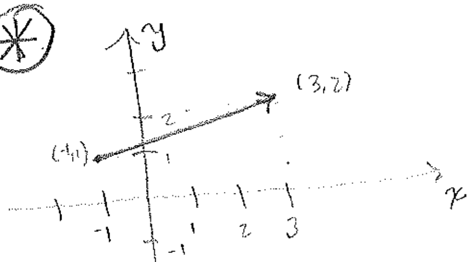
(b) $\vec{a}-\vec{b}$

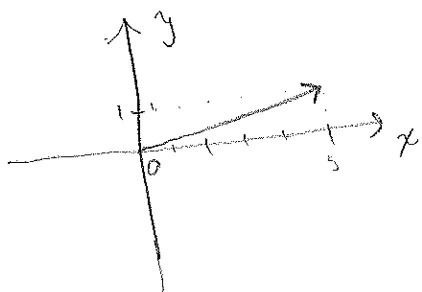


(c)



(9) ✳





(19) ✳ $\vec{a}=\langle 5,-12\rangle$, $\vec{b}=\langle -3,-6\rangle$

$\vec{a}+\vec{b}=\langle 5,-12\rangle+\langle -3,-6\rangle=\langle 2,-18\rangle$

$2\vec{a}+3\vec{b}=2\langle 5,-12\rangle+3\langle -3,-6\rangle=\langle 10,-24\rangle+\langle -9,-18\rangle=\langle 1,-42\rangle$

$|\vec{a}|=\sqrt{5^2+12^2}=\sqrt{25+144}=\sqrt{169}=13$

$|\vec{a}-\vec{b}|=|\langle 5,-12\rangle-\langle -3,-6\rangle|=|\langle 8,-6\rangle|=\sqrt{8^2+6^2}=\sqrt{64+36}=\sqrt{100}=10$

(20) $\vec{a} = 4\hat{i} + \hat{j}$, $\vec{b} = \hat{i} - 2\hat{j}$

(*)

$\vec{a} + \vec{b} = (4\hat{i} + \hat{j}) + (\hat{i} - 2\hat{j}) = 4\hat{i} + \hat{i} + \hat{j} - 2\hat{j} = 5\hat{i} - \hat{j}$

$2\vec{a} + 3\vec{b} = 2(4\hat{i} + \hat{j}) + 3(\hat{i} - 2\hat{j}) = 8\hat{i} + 2\hat{j} + 3\hat{i} - 6\hat{j} = 11\hat{i} - 4\hat{j}$

$|\vec{a}| = \sqrt{4^2 + 1^2} = \sqrt{16+1} = \sqrt{17}$

$|\vec{a} - \vec{b}| = |(4\hat{i} + \hat{j}) - (\hat{i} - 2\hat{j})| = |4\hat{i} - \hat{i} + \hat{j} + 2\hat{j}| = |3\hat{i} + 3\hat{j}| = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$

(21) $\vec{a} = \hat{i} + 2\hat{j} - 3\vec{k}$, $\vec{b} = -2\hat{i} - \hat{j} + 5\hat{k}$

$\vec{a} + \vec{b} = (\hat{i} + 2\hat{j} - 3\hat{k}) + (-2\hat{i} - \hat{j} + 5\hat{k}) = \hat{i} - 2\hat{i} + 2\hat{j} - \hat{j} - 3\hat{k} + 5\hat{k} = -\hat{i} + \hat{j} + 2\hat{k}$

$2\vec{a} + 3\vec{b} = 2(\hat{i} + 2\hat{j} - 3\hat{k}) + 3(-2\hat{i} - \hat{j} + 5\hat{k}) = 2\hat{i} + 4\hat{j} - 6\hat{k} - 6\hat{i} - 3\hat{j} + 15\hat{k} = -4\hat{i} + \hat{j} + 9\hat{k}$

$|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$

$|\vec{a} - \vec{b}| = |(\hat{i} + 2\hat{j} - 3\hat{k}) - (-2\hat{i} - \hat{j} + 5\hat{k})| = |\hat{i} + 2\hat{i} + 2\hat{j} + \hat{j} - 3\hat{k} - 5\hat{k}| = |3\hat{i} + 3\hat{j} - 8\hat{k}|$

$= \sqrt{3^2 + 3^2 + 8^2} = \sqrt{9+9+64} = \sqrt{18+64} = \sqrt{82}$

(*)

(23) Find a unit vector that has the same direction as the given vector.

Let $\vec{v} = -3\hat{i} + 7\hat{j}$. Then $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$ is the desired vector. Indeed,

$\vec{u} = \frac{-3\hat{i} + 7\hat{j}}{\sqrt{3^2 + 7^2}} = \frac{-3\hat{i} + 7\hat{j}}{\sqrt{9+49}} = \frac{-3\hat{i} + 7\hat{j}}{\sqrt{58}} = \frac{-3}{\sqrt{58}}\hat{i} + \frac{7}{\sqrt{58}}\hat{j}$

(*) → NOTE: DO SIMILAR EXERCISE

(26) Find a vector that has the same direction as $\langle -2, 4, 2 \rangle$ but has length 6.

Let $\vec{v}$ be such a vector. Then, $\vec{v} = \frac{6 \cdot \langle -2,4,2 \rangle}{\sqrt{2^2 + 4^2 + 2^2}} = \frac{6 \cdot \langle -2,4,2 \rangle}{\sqrt{4+16+4}}$

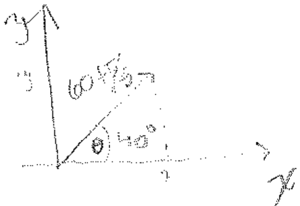
$= \frac{6 \cdot \langle -2,4,2 \rangle}{\sqrt{24}} = \frac{6}{2\sqrt{6}} \langle -2,4,2 \rangle = \frac{3}{\sqrt{6}} \langle -2,4,2 \rangle$.

We can check that indeed this is the desired vector

$|\vec{v}| = \left| \frac{3}{\sqrt{6}} \langle -2,4,2 \rangle \right| = \sqrt{\left(\frac{6}{\sqrt{6}}\right)^2 + \left(\frac{12}{\sqrt{6}}\right)^2 + \left(\frac{6}{\sqrt{6}}\right)^2} = \sqrt{\frac{36}{6} + \frac{144}{6} + \frac{36}{6}} \quad \sqrt{\frac{216}{6}}$

$= \sqrt{36} = 6$.

(31)

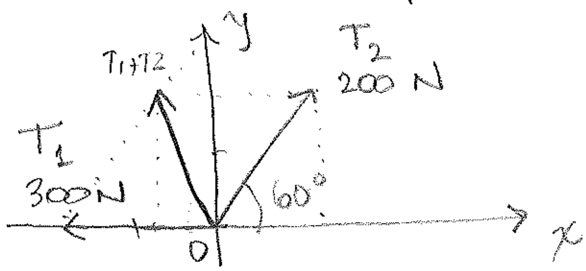


$\sin\theta = \frac{op}{H}$, $\cos\theta = \frac{adj}{H}$

$\sin 40° = \frac{y}{60} \Rightarrow y = \sin(40°) \times 60$ ft/s

$\cos 40° = \frac{x}{60} \Rightarrow x = \cos(40°) \times 60$ ft/s

(33) Find the magnitude of the resultant force and the angle it makes with the positive x-axis.



$\vec{T_1} = \langle -300, 0 \rangle$

$\cos 60° = \dfrac{T_{2x}}{|T_2|} \Rightarrow T_{2x} = \cos 60° \cdot 200$

$T_{2y} = \sin 60° \cdot 200$

$\vec{T_2} = \langle 200 \cdot \cos 60°, 200 \cdot \sin 60° \rangle$

$= 200 \langle \cos 60°, \sin 60° \rangle$

$= 200 \langle 1/2, \frac{\sqrt{3}}{2} \rangle$

$= 100 \langle 1, \sqrt{3} \rangle$

$\vec{T_1} + \vec{T_2} = \langle -300, 0 \rangle + 100 \langle 1, \sqrt{3} \rangle$

$= \langle -200, 100\sqrt{3} \rangle$

$|\vec{T_1} + \vec{T_2}| = \sqrt{(-200)^2 + (100\sqrt{3})^2} = \sqrt{40000 + 30000} = \sqrt{70000}$

$= 100\sqrt{7}$

the resulting magnitude is $\boxed{100\sqrt{7}\ N}$

to find the angle, we compute

$\cos \theta° = \dfrac{-200}{100\sqrt{7}} \Rightarrow \theta° = \arccos\left(\dfrac{-2\cancel{00}}{1\cancel{00}\sqrt{7}}\right) = \arccos\left(\dfrac{-2}{\sqrt{7}}\right) = 2{,}427$ rad

$\Rightarrow \boxed{\theta° \simeq 139.106 \text{ degrees}}$

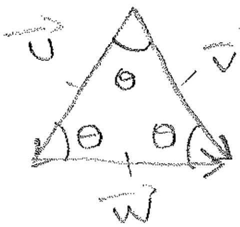
Section 12.3 :

(9) Find $\vec{a} \cdot \vec{b}$

$|\vec{a}| = 6$, $|\vec{b}| = 5$ the angle between $\vec{a}$ and $\vec{b}$ is $2\pi/3$

$$\Rightarrow \quad \vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta^\circ$$
$$= 6 \cdot 5 \cdot \cos\left(2\pi/3\right) = 30 \cdot \left(-\frac{1}{2}\right) = \boxed{-15}$$

(11) If $\vec{u}$ is a unit vector, find $\vec{u} \cdot \vec{v}$ and $\vec{u} \cdot \vec{w}$.



Since all three sides of this triangle are the same we know that $\theta = 60^\circ$

Hence :

$$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$$
$$= 1 \cdot 1 \cdot \cos(60^\circ) = \frac{1}{2}$$

Now, for

$$\vec{u} \cdot \vec{w} = |\vec{u}| \cdot |\vec{w}| \cdot \cos\theta = 1 \cdot 1 \cdot \cos 120^\circ = -\frac{1}{2}$$

(13) Show that $\vec{i} \cdot \hat{j} = \hat{j} \cdot \vec{k} = \hat{k} \cdot \hat{i} = 0$

(a) $\vec{i} \cdot \hat{j} = \langle 1,0,0\rangle \cdot \langle 0,1,0\rangle = 0+0+0 = 0 = 1 \cdot 1 \cdot \cos 90^\circ$

$\hat{j} \cdot \vec{k} = \langle 0,1,0\rangle \cdot \langle 0,0,1\rangle = 0+0+0 = 0 = 1 \cdot 1 \cdot \cos 90^\circ$

$\vec{k} \cdot \hat{i} = \langle 0,0,1\rangle \cdot \langle 1,0,0\rangle = 0+0+0 = 0 = 1 \cdot 1 \cdot \cos 90^\circ$

(b) $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

$\hat{i} \cdot \hat{i} = \langle 1,0,0\rangle \cdot \langle 1,0,0\rangle = 1+0+0 = 1 = 1 \cdot 1 \cdot \cos 0^\circ$

$\hat{j} \cdot \hat{j} = \langle 0,1,0\rangle \cdot \langle 0,1,0\rangle = 0+1+0 = 1 = 1 \cdot 1 \cdot \cos 0^\circ$

$\hat{k} \cdot \vec{k} = \langle 0,0,1\rangle \cdot \langle 0,0,1\rangle = 0+0+1 = 1 = 1 \cdot 1 \cdot \cos 0^\circ$

14) A street vendor sells a hamburgers, b hot dogs, and c soft drinks on a given day. He charges \$2 for a hamburger, \$1.50 for a hot dog, and \$1 for a soft drink. If $A = \langle a, b, c\rangle$ and $P = \langle 2, 1.5, 1\rangle$, what is the meaning of the dot product $A \cdot P$?

By definition $A \cdot P = \langle a,b,c\rangle \cdot \langle 2,1.5,1\rangle = a \cdot 2 + b \cdot 1.5 + c \cdot 1$ = Total amount of sales on a given day

(15) Find the angle between the vectors:

$\vec{a} = \langle 4,3 \rangle$, $\vec{b} = \langle 2,-1 \rangle$

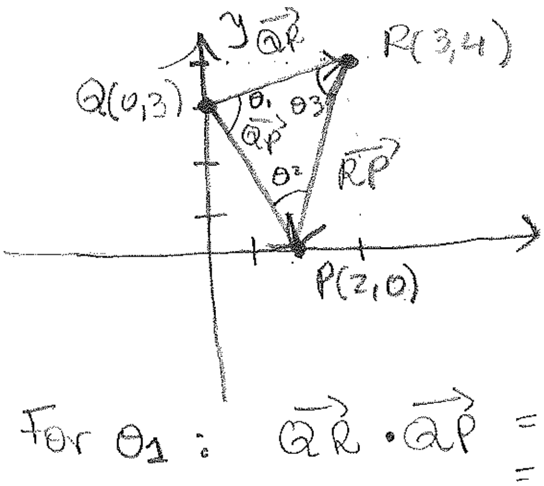
$$\vec{a}\cdot\vec{b} = \langle 4,3\rangle \cdot \langle 2,-1 \rangle = 8 + (-3) = 5$$
$$= |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta$$
$$= \sqrt{16+9}\,\sqrt{4+1}\cdot\cos\theta$$
$$= \sqrt{25}\,\sqrt{5}\cdot\cos\theta$$
$$= 5\sqrt{5}\cdot\cos\theta^\circ \Rightarrow 5 = 5\sqrt{5}\cdot\cos\theta^\circ$$
$$\Rightarrow \frac{1}{\sqrt{5}} = \cos\theta^\circ \Rightarrow \theta = \arccos\left(\frac{1}{\sqrt{5}}\right) \simeq 63.4^\circ$$

(19) Find the angle between the vectors:

$\vec{a} = 4\hat{\imath} - 3\hat{\jmath} + \hat{k}$, $\vec{b} = 2\hat{\imath} - \hat{k}$

$$\vec{a}\cdot\vec{b} = \langle 4,-3,1\rangle \cdot \langle 2,0,-1\rangle = 8 + 0 - 1 = 7$$
$$= |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta$$
$$= \sqrt{16+9+1}\,\sqrt{4+1}\cdot\cos\theta$$
$$= \sqrt{26}\,\sqrt{5}\cdot\cos\theta \Rightarrow 7 = \sqrt{130}\,\cos\theta$$
$$\Rightarrow \theta = \arccos\left(\frac{7}{\sqrt{130}}\right) \simeq 52^\circ$$

(21) Find the three angles of the triangle with the given vertices



$$\vec{QR} = (3,4) - (0,3) = \langle 3,1 \rangle$$
$$\vec{QP} = (2,0) - (0,3) = \langle 2,-3 \rangle$$
$$\vec{RP} = (2,0) - (3,4) = \langle -1,-4 \rangle$$

To obtain the angles:

For θ_1: $\vec{QR}\cdot\vec{QP} = \langle 3,1\rangle \cdot \langle 2,-3\rangle = 6 - 3 = 3$
$$= \sqrt{9+1}\cdot\sqrt{4+9}\cdot\cos\theta = \sqrt{130}\cdot\cos\theta_1$$
$$\Rightarrow \theta_1 = \arccos\left(\frac{3}{\ }\right) \simeq 75^\circ$$

For θ_3: $(-1 \cdot \overrightarrow{QR}) \cdot (\overrightarrow{RP}) =$

$(-1 \cdot \langle 3,1 \rangle) \cdot \langle -1,-4 \rangle = \langle -3,-1 \rangle \cdot \langle -1,-4 \rangle = 3+4 = 7$

$= \sqrt{9+1}\sqrt{1+16} \cdot \cos\theta$

$= \sqrt{170} \cdot \cos\theta°$

$\Rightarrow \quad \theta = \arccos\left(\frac{7}{\sqrt{170}}\right) \approx 57°$

Now, for θ_2 we have two options:

(1) Since we know that the angles of a triangle must add up to $180°$, we have the equation

$180° = \theta_1 + \theta_2 + \theta_3 = 75 + \theta_2 + 57$

$\Rightarrow \quad \theta_2 = 180 - 75 - 57 = 48°$

(2) Calculate as we did for θ_1 and θ_3,

$((-1)\overrightarrow{QP}) \cdot (-1)\overrightarrow{RP} = ((-1)\langle 2,-3 \rangle) \cdot ((-1) \cdot \langle -1,-4 \rangle)$

$= \langle -2,3 \rangle \cdot \langle 1,4 \rangle = -2 + 12 = 10$

$= \sqrt{4+9}\sqrt{1+16} \cdot \cos\theta_2$

$= \sqrt{13}\sqrt{17} \cdot \cos\theta_2$

$\Rightarrow \quad \theta_2° = \arccos\left(\frac{10}{\sqrt{221}}\right) \approx 48°$

(23) Determine whether the given vectors are orthogonal, parallel, or neither.

(a) $\vec{a} = \langle -5, 3, 7 \rangle$, $\vec{b} = \langle 6, -8, 2 \rangle$

$\vec{a} \cdot \vec{b} = \langle -5,3,7 \rangle \cdot \langle 6,-8,2 \rangle = -30 - 24 + 14 = -30 - 10 = -40$

$|\vec{a}| = \sqrt{25+9+49} = \sqrt{83}$ $\quad |\vec{a}| \cdot |\vec{b}| = \sqrt{83 \times 104} = 92{,}9$

$|\vec{b}| = \sqrt{36+64+4} = \sqrt{104}$

Hence, these vectors are neither orthogonal nor parallel.

(b) $\vec{a} = \langle 4, 6 \rangle$, $\vec{b} = \langle -3, 2 \rangle$

$\vec{a} \cdot \vec{b} = \langle 4,6 \rangle \cdot \langle -3,2 \rangle = -12 + 12 = 0 \Rightarrow$ $\vec{a}$ is orthogonal to $\vec{b}$

(c) $\vec{a} = -\hat{\imath} + 2\hat{\jmath} + 5\hat{k}$, $\vec{b} = 3\hat{\imath} + 4\hat{\jmath} - \hat{k}$

$\vec{a} \cdot \vec{b} = \langle -1,2,5 \rangle \cdot \langle 3,4,-1 \rangle = -3 + 8 - 5 = 0 \Rightarrow \vec{a} \perp \vec{b}$

(d) $\vec{a} = 2\hat{\imath} + 6\hat{\jmath} - 4\hat{k}$, $b = -3\hat{\imath} - 9\hat{\jmath} + 6\hat{k}$

$\vec{a} \cdot \vec{b} = \langle 2,6,-4 \rangle \cdot \langle -3,-9,6 \rangle = -6 - 54 - 24 = -84$

$|\vec{a}| = \sqrt{4+36+16} = \sqrt{56}$

$|\vec{b}| = \sqrt{9+81+36} = \sqrt{126}$

$\Big\} \Rightarrow |\vec{a}| \cdot |\vec{b}| = \sqrt{56 \times 126} = 84$

Hence, these vectors are parallel

(25) Use vectors to decide whether the triangle with vertices $P(1,-3,-2)$, $Q(2,0,-4)$, and $R(6,-2,-5)$ is right-angled.

$$\vec{PQ} = (2,0,-4) - (1,-3,-2) = \langle 1, 3, -2 \rangle$$
$$\vec{PR} = (6,-2,-5) - (1,-3,-2) = \langle 5, 1, -3 \rangle$$
$$\vec{RQ} = (2,0,-4) - (6,-2,-5) = \langle -4, 2, 1 \rangle$$

$$\vec{PQ} \cdot \vec{PR} = \langle 1,3,-2\rangle \cdot \langle 5,1,-3\rangle = 5+3+6 > 0$$
$$\vec{PQ} \cdot \vec{RQ} = \langle 1,3,-2\rangle \cdot \langle -4,2,1\rangle = -4+6-2 = 0$$

$\Rightarrow$ the vector $\vec{PQ}$ and $\vec{RQ}$ are orthogonal

$\Rightarrow$ the triangle is right-angled

(26) Find the values of x such that the angle between the vectors $\langle 2,1,-1\rangle$ and $\langle 1,x,0\rangle$ is $45°$.

$$\langle 2,1,-1\rangle \cdot \langle 1,x,0\rangle = 2+x+0 = 2+x$$
$$= \sqrt{4+1+1} \cdot \sqrt{1+x^2} \cdot \cos 45°$$

$$\Rightarrow \quad 2+x = \sqrt{6+6x^2} \cdot \cos 45°$$

$$2+x = \sqrt{6+6x^2} \cdot \frac{\sqrt{2}}{2}$$

$$4+2x = \sqrt{12+12x^2} \quad \Rightarrow \quad 16+16x+4x^2 = 12+12x^2$$

$$\Rightarrow \quad 8x^2-16x-4=0 \quad \Leftrightarrow \quad 2x^2-4x-1=0$$

$$x = \frac{4 \pm \sqrt{16+8}}{4} = \frac{4 \pm \sqrt{24}}{4} = \frac{4 \pm 2\sqrt{6}}{4} = \boxed{1 \pm \frac{1}{2}\sqrt{6}}$$

(27) Find a unit vector that is orthogonal to both

$\hat{i}+\hat{j}$ and $\hat{i}+\hat{k}$.

Let $\vec{v}$ be a vector orthogonal to these. then:

$$(\hat{i}+\hat{j})\cdot(\vec{v})=0=\langle 1,1,0\rangle\cdot\langle v_1,v_2,v_3\rangle=v_1+v_2$$
$$(\hat{i}+\hat{k})\cdot(\vec{v})=0=\langle 1,0,1\rangle\cdot\langle v_1,v_2,v_3\rangle=v_1+v_3$$

$\Rightarrow$ $v_1+v_2=0$, $v_1+v_3=0$. A possible vector is $\vec{v}=\langle 1,-1,-1\rangle$. However, this is not a unit vector. We would have to normalize it

$$\hat{v}=\frac{\vec{v}}{|\vec{v}|}=\frac{\langle 1,-1,-1\rangle}{\sqrt{1^2+1^2+1^2}}\Rightarrow \boxed{\hat{v}=\frac{1}{\sqrt{3}}\langle 1,-1,-1\rangle}$$

(28) Find two unit vectors that make an angle of $60°$ with $\vec{v}=\langle 3,4\rangle$.

Let $\vec{u_1}$ be such vector. then

$$\vec{v}\cdot\vec{u_1}=|\vec{v}|\cdot 1\cdot\cos 60°=\frac{5}{2}$$
$$=\langle 3,4\rangle\cdot\langle u_1,u_2\rangle=3u_1+4u_2$$

$$\Rightarrow \frac{5}{2}=3u_1+4u_2 \Rightarrow \frac{5}{2}-3u_1=4u_2 \Rightarrow u_2=\frac{5}{8}-\frac{3}{4}u_1$$

<u>AND</u>: $|\vec{u}|=\sqrt{u_1^2+u_2^2}=1 \Rightarrow u_1^2+u_2^2=1$.

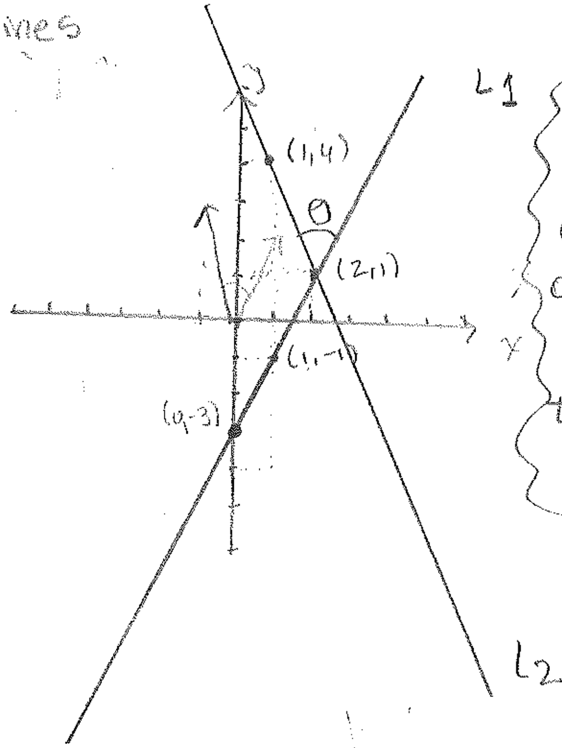
$$\Rightarrow u_1^2+\left(\frac{5}{8}-\frac{3}{4}u_1\right)^2=1$$

$$u_1^2+\frac{25}{64}-\frac{30}{32}u_1+\frac{9}{16}u_1^2=1$$

2.9 Find the acute angle between the lines

$2x - y = 3$, $3x + y = 7$

$y = 2x - 3$, $y = 7 - 3x$



Any line in $\mathbb{R}^2$ Euclidean Space $ax + by + c = 0$ One direction vector is $t(-b, a)$ $t \in \mathbb{R}$.

the direction vectors are:

for L1: $\vec{b_1} = \langle 1, 2 \rangle$

for L2: $\vec{b_2} = \langle -1, 3 \rangle$

the acute angle between the lines can be found using the dot product:

$$\vec{b_1} \cdot \vec{b_2} = \langle 1,2 \rangle \cdot \langle -1,3 \rangle$$
$$= -1 + 6$$
$$= \boxed{5}$$

Also,

Acute angle: angle less than 90°

$$\vec{b_1} \cdot \vec{b_2} = |\vec{b_1}| \cdot |\vec{b_2}| \cdot \cos\theta$$
$$= \sqrt{4+1}\sqrt{9+1} \cdot \cos\theta$$
$$= \sqrt{50} \cos\theta$$

Hence, $5 = \sqrt{50} \cdot \cos\theta \implies \theta = \arccos\left(\frac{5}{\sqrt{50}}\right) = \boxed{45°}$

NOTE: To find the direction vector of a line in 2-D, we can use $(1, m)$, where m is the slope

Do this problem, but change the equations to

$x - y = 3$ $2x + y = 4$

Find the acute angle between the lines

$L_1: \quad x - y = 3 \qquad L_2: 2x + y = 4$

$y = x - 3 \qquad y = -2x + 4$

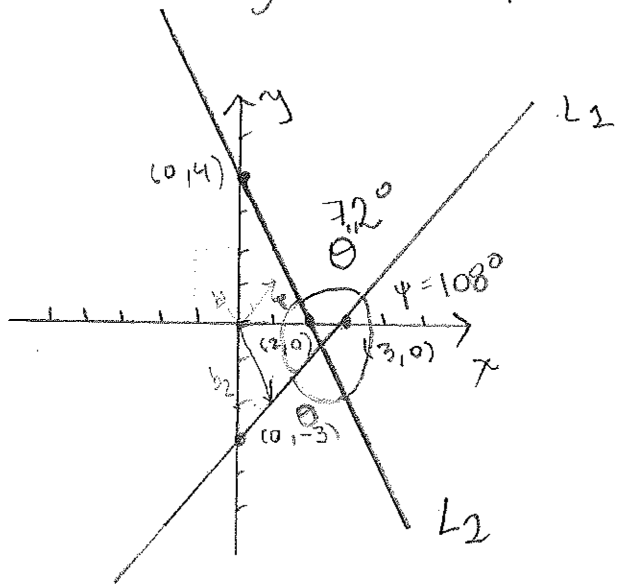
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

For L_1:

$(x_1, y_1) = (1, -2)$

$(x_2, y_2) = (2, -1)$

$$\Rightarrow m = \frac{-1-(-2)}{2-1} = 1$$



the direction vectors are

for L_1: $\vec{b_1} = \langle 1, 1 \rangle$

for L_2: $\vec{b_2} = \langle 1, -2 \rangle$ I can work with any scalar multiple.

so now $\vec{b_2} = (-1)\langle 1, -2 \rangle = \langle -1, 2 \rangle$

the acute angle between the lines is the acute angle between their direction vectors:

$$\vec{b_1} \cdot \vec{b_2} = \langle 1, 1 \rangle \cdot \langle 1, -2 \rangle = 1 - 2 = -1$$

Also, $\vec{b_1} \cdot \vec{b_2} = |\vec{b_1}| \cdot |\vec{b_2}| \cdot \cos\theta$

$= \sqrt{1+1} \cdot \sqrt{1+4} \cdot \cos\theta$

$= \sqrt{10} \cdot \cos\theta$

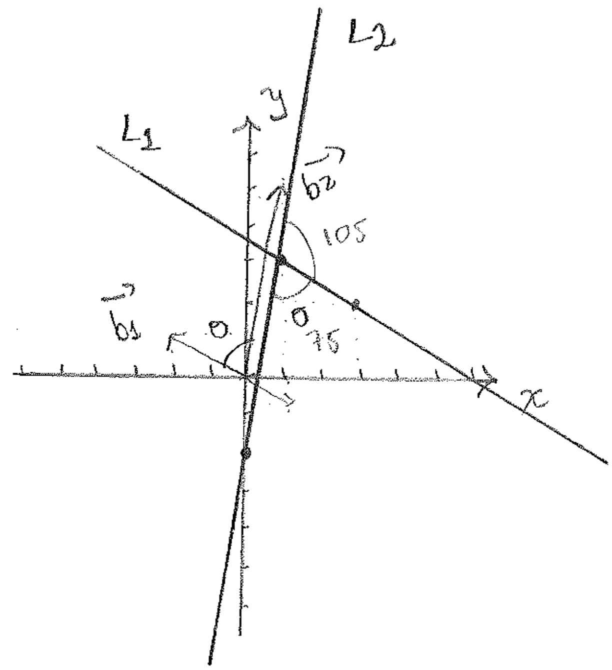
Hence, $-1 = \sqrt{10}\cos\theta \Rightarrow \theta = \arccos\left(\frac{-1}{\sqrt{10}}\right) \simeq 108$

The acute angle is $180 - 108 \simeq 72$

(30) Find the acute angle between the lines (8)

$L_1 := x + 2y = 7 \qquad L_2 := 5x - y = 2$

the direction vectors are of the form $(-b, a)$, where the line is given as $ax + by + c = 0$



For $L_1 :=$ $\vec{b_1} = \langle -2, 1 \rangle$

For $L_2 :=$ $\vec{b_2} = \langle 1, 5 \rangle$

$$\vec{b_1} \cdot \vec{b_2} = \langle -2, 1 \rangle \cdot \langle 1, 5 \rangle = -2 + 5 = 3$$

$$\vec{b_1} \cdot \vec{b_2} = |\vec{b_1}| \cdot |\vec{b_2}| . \cos\theta = \sqrt{4+1}\sqrt{1+25} \cdot \cos\theta = \sqrt{5 \times 26} \cdot \cos\theta$$

$$\Rightarrow \theta = \arccos\left(\frac{3}{\sqrt{5 \times 26}}\right) = 75^\circ$$

$(1, 3)$, $(7, 0)$
$x_1 \; y_1 \quad x_2 \; y_2$

$$\left(1, \frac{y_2 - y_1}{x_2 - x_1}\right) = \left(1, \frac{0-3}{7-1}\right) = \left(1, \frac{-3}{6}\right) = \left(1, \frac{-1}{2}\right)$$

$$\left(1, \frac{y_2 - y_1}{x_2 - x_1}\right) = \left(1, \frac{-2-3}{0-1}\right) = \left(1, \frac{-5}{-1}\right) = (1, 5)$$

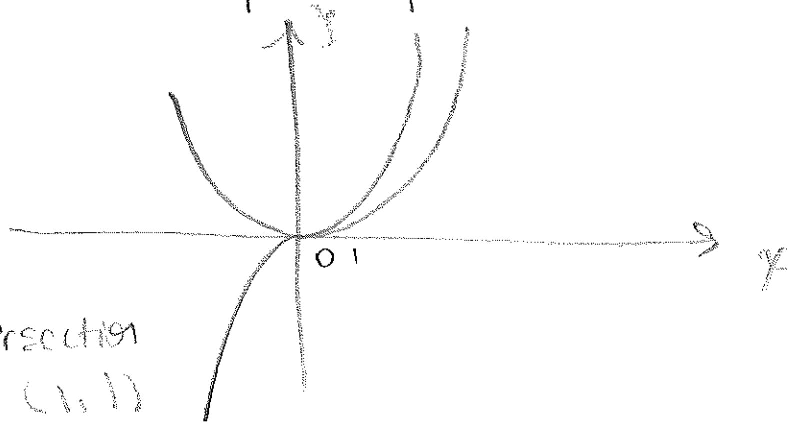
$\langle 1, \frac{-1}{2} \rangle \cdot \langle 1, 5 \rangle = 1 - \frac{5}{2} = -\frac{3}{2}$ $\qquad (1, 3)$, $(0, -2)$
$x_1 \; y_1 \quad x_2 \; y_2$

$$\sqrt{1 + \frac{1}{4}} \sqrt{1+25} \cos\theta$$

$$\sqrt{\frac{5}{4} \cdot 26} \cos\theta \Rightarrow \theta = \arccos\left(\frac{-\frac{3}{2}}{\sqrt{\frac{5}{4} \cdot 26}}\right) \approx 105.$$

31 Find the acute angles between the curves at their points of intersection. (The angle between two curves is the angle between their tangent lines at the point of intersection).

$y = x^2$
$y = x^3$



the points of intersection are (0,0) and (1,1)

For (0,0).
the slope of the tangent line for $y=x^2$ at (0,0) is :

$$y'(0) = 2(0) = 0.$$

the line is $y_1 = 0x + 0 \Rightarrow \boxed{y_1 = 0}$

Likewise for $y = x^3$ at (0,0)

$$y'(0) = 3(0)^2 = 0$$

the line is $y_2 = 0x + 0 \Rightarrow \boxed{y_2 = 0}$

Hence, at (0,0) the lines are the same and the angle is $0°$

For (1,1).
the slope of the tangent line for $y = x^2$ at (1,1) is:

$$y'(1) = 2(1) = 2.$$

the line is $y_1 = 2x + b$. A point on the line we know is (1,1). Hence $1 = 2(1) + b \Rightarrow b = -1$

$\Rightarrow \boxed{y_1 = 2x - 1}$

Likewise for $y = x^3$ at (1,1)

$$y'(1) = 3(1)^2 = 3$$

the line is $y_2 = 3x + b$. But $1 = 3(1) + b \Rightarrow b = -2$

$\Rightarrow \boxed{y_2 = 3x - 2}$

Now we find the angle like we did before:

$y_1 = 2x - 1 \iff -2x + y_1 + 1 = 0$

$\Rightarrow \vec{b_1} = \langle -b, a \rangle = \langle -1, -2 \rangle$

$y_2 = 3x - 2 \iff -3x + y_2 + 2 = 0$

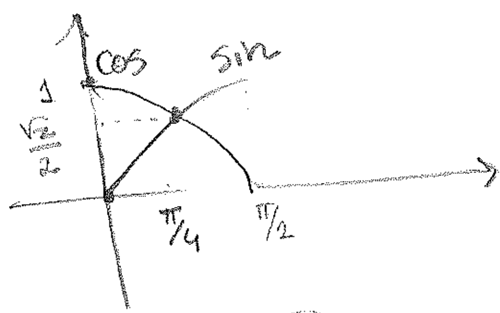
$\Rightarrow \vec{b_2} = \langle -b, a \rangle = \langle -1, -3 \rangle$

Hence, $\vec{b_1} \cdot \vec{b_2} = \langle -1, -2 \rangle \cdot \langle -1, -3 \rangle = 1 + 6 = 7$

$$\vec{b_1} \cdot \vec{b_2} = |\vec{b_1}| \cdot |\vec{b_2}| \cdot \cos\theta^\circ$$
$$= \sqrt{1+4}\sqrt{1+9} \cdot \cos\theta^\circ$$
$$= \sqrt{50} \cdot \cos\theta^\circ$$

$$\Rightarrow \theta = \arccos\left(\frac{7}{\sqrt{50}}\right) = 8.1^\circ$$

(32) $y = \sin x$, $y = \cos x$, $0 \le x \le \pi/2$



If $x = \frac{\pi}{4}$, then $\sin x = \cos x$

For $y = \sin x$: $y'(\pi/4) = \cos(\pi/4) = \frac{\sqrt{2}}{2}$. $y_1 = \frac{\sqrt{2}}{2}x + b$

$$\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\frac{\pi}{4} + b \Rightarrow \frac{\sqrt{2}}{2} - \frac{\pi}{4}\frac{\sqrt{2}}{2} = b \Rightarrow \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right) = b$$

$$\Rightarrow \frac{\sqrt{2}}{2}\,\frac{4-\pi}{4} = b = \frac{4\sqrt{2} - \pi\sqrt{2}}{8} = b$$

For $y = \cos x$:

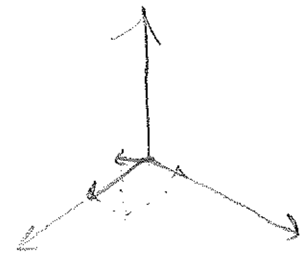
$y'(\pi/4) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$. $y_2 = -\frac{\sqrt{2}}{2}x + b$

$$\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\frac{\pi}{4} + b \Rightarrow \frac{\sqrt{2}}{2} + \frac{\pi}{4}\frac{\sqrt{2}}{2} = b \Rightarrow \frac{\sqrt{2}}{2}\left(1 + \frac{\pi}{4}\right) = b$$

$$\Rightarrow \frac{\sqrt{2}}{2}\,\frac{4+\pi}{4} = b = \frac{4\sqrt{2} + \pi\sqrt{2}}{8} = b$$

(33) Find the direction cosines and direction angles of the vector.

$\vec{v} = \langle 2,1,2 \rangle \qquad |\vec{v}| = \sqrt{4+1+4} = \sqrt{9} = 3$



$\cos\alpha = \frac{2}{3}, \quad \cos\beta = \frac{1}{3}, \quad \cos\gamma = \frac{2}{3}$

$\alpha = \arccos\left(\frac{2}{3}\right), \quad \beta = \arccos\left(\frac{1}{3}\right), \quad \gamma = \arccos\left(\frac{2}{3}\right)$

$\alpha = 48^\circ, \quad \beta = 70^\circ, \quad \gamma = 48^\circ$

(39) Find the scalar and vector projections of $\vec{b}$ onto $\vec{a}$

$\vec{a} = \langle -5,12 \rangle, \quad \vec{b} = \langle 4,6 \rangle$

$$\text{Comp}_a b = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{\langle -5,12\rangle \cdot \langle 4,6\rangle}{\sqrt{25+144}} = \frac{-20+72}{\sqrt{169}} = \frac{52}{13} = \boxed{4}$$

$$\text{Proj}_a b = \text{Comp}_a b \frac{\vec{a}}{|\vec{a}|} = 4 \cdot \frac{\langle -5,12\rangle}{13} = \boxed{\frac{4}{13} \cdot \langle -5,12 \rangle}$$

(41) Same as before:

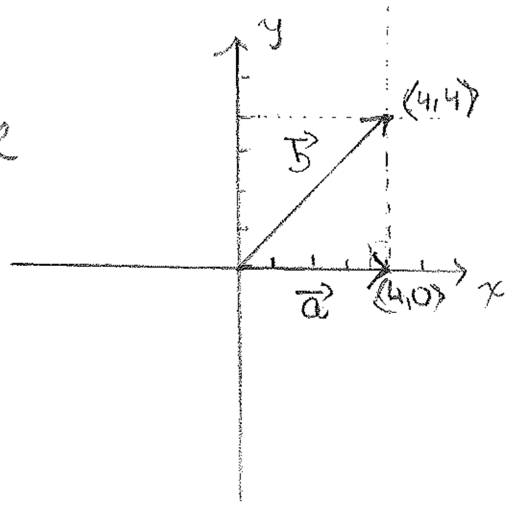
$\vec{a} = \langle 3,6,-2 \rangle, \quad \vec{b} = \langle 1,2,3 \rangle$

$$\text{Comp}_a b = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{\langle 3,6,-2\rangle \cdot \langle 1,2,3\rangle}{\sqrt{9+36+4}} = \frac{3+12-6}{\sqrt{49}} = \boxed{\frac{9}{7}}$$

$$\text{Proj}_a b = \text{Comp}_a b \frac{\vec{a}}{|\vec{a}|} = \boxed{\frac{9}{7^2} \cdot \langle 3,6,-2 \rangle}$$

Similar to Exercises 39 - 44

Simple Example



$$\text{Comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{\langle 4,4\rangle \cdot \langle 4,0\rangle}{\sqrt{4^2}} = \frac{16}{4} = \boxed{4}$$

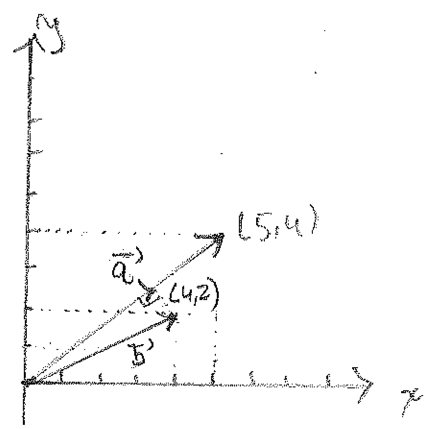
$$\text{Proj}_{\vec{a}}\vec{b} = \text{Comp}_{\vec{a}}\vec{b}\cdot\frac{\vec{a}}{|\vec{a}|} = \frac{4}{4}\cdot\vec{a} = \vec{a}$$

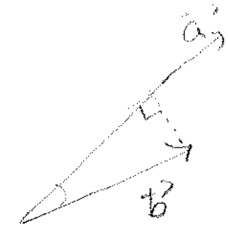
A more complicated

Let $\vec{a} = \langle 5,4\rangle$

$\vec{b} = \langle 4,2\rangle$.

We want the projection of $\vec{b}$ onto $\vec{a}$





By trigonometry:

$$\cos\theta = \frac{\text{Comp}_{\vec{a}}\vec{b}}{|\vec{b}|}$$

$$\Rightarrow \text{Comp}_{\vec{a}}\vec{b} = \cos\theta\,|\vec{b}|$$

By Dot product:

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta$$

$$\Rightarrow |\vec{b}|\cdot\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

Replacing this into our $\text{Comp}_{\vec{a}}\vec{b}$ definition: $\text{Comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

In our example: $\text{Comp}_{\vec{a}}\vec{b} = \dfrac{\langle 5,4\rangle\cdot\langle 4,2\rangle}{\sqrt{25+16}} = \dfrac{20+8}{\sqrt{41}} = \dfrac{28}{\sqrt{41}}$

Now, to get the vector projection, we can just multiply this by a unit vector in $\vec{a}$'s direction: $\text{Proj}_{\vec{a}}\vec{b} = \text{Comp}_{\vec{a}}\vec{b}\,\dfrac{\vec{a}}{|\vec{a}|} = \dfrac{28}{41}\langle 5,4\rangle$

(45) Show that the vector $\text{orth}_{\vec{a}}\vec{b} = b - \text{proj}_{\vec{a}}\vec{b}$ is orthogonal to $\vec{a}$

Two vectors $\vec{a}, \vec{b}$ are orthogonal iff $\vec{a} \cdot \vec{b} = 0$.

In this case: (For simplicity, let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$)

$$\vec{a} \cdot \text{orth}_{\vec{a}}\vec{b} = \vec{a} \cdot (\vec{b} - \text{proj}_{\vec{a}}\vec{b})$$

By definition of projection:

$$\text{orth}_{\vec{a}}\vec{b} = \vec{b} - \text{proj}_{\vec{a}}\vec{b}$$

$$= \langle b_1, b_2, b_3 \rangle - \frac{\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle}{a_1^2 + a_2^2 + a_3^2} \cdot \langle a_1, a_2, a_3 \rangle$$

$$= \langle b_1, b_2, b_3 \rangle - \frac{a_1b_1 + a_2b_2 + a_3b_3}{a_1^2 + a_2^2 + a_3^2} \cdot \langle a_1, a_2, a_3 \rangle$$

Using this vector, compute:

$$\langle a_1, a_2, a_3 \rangle \cdot \left(\langle b_1, b_2, b_3 \rangle - \frac{a_1b_1 + a_2b_2 + a_3b_3}{a_1^2 + a_2^2 + a_3^2} \cdot \langle a_1, a_2, a_3 \rangle \right)$$

$$= a_1 \cdot \left(b_1 - \left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{a_1^2 + a_2^2 + a_3^2} \right) a_1 \right) + a_2 \cdot \left(b_2 - \left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{a_1^2 + a_2^2 + a_3^2} \right) a_2 \right)$$
$$+ a_3 \cdot \left(b_3 - \left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{a_1^2 + a_2^2 + a_3^2} \right) a_3 \right)$$

$$= a_1 \cdot \left(\frac{a_1^2b_1 + a_2^2b_1 + a_3^2b_1 - a_1^2b_1 - a_1a_2b_2 - a_1a_3b_3}{a_1^2 + a_2^2 + a_3^2} \right)$$
$$+ a_2 \cdot \left(\frac{a_1^2b_2 + a_2^2b_2 + a_3^2b_2 - a_1a_2b_1 - a_2^2b_2 - a_2a_3b_3}{a_1^2 + a_2^2 + a_3^2} \right)$$
$$+ a_3 \cdot \left(\frac{a_1^2b_3 + a_2^2b_3 + a_3^2b_3 - a_1a_3b_1 - a_2a_3b_2 - a_3^2b_3}{a_1^2 + a_2^2 + a_3^2} \right)$$

$$= \cancel{a_1^3 b_1} + \cancel{a_1 a_2^2 b_1} + a_1 a_3^2 b_1 - \cancel{a_1^3 b_1} - \cancel{a_1^2 a_2 b_2} - \cancel{a_1^2 a_3 b_3} + \cancel{a_1^2 a_2 b_2} + \cancel{a_2^3 b_2} + a_2 a_3^2 b_2$$

$$a_1^2 + a_2^2 + a_3^2$$

$$- \cancel{a_1 a_2^2 b_1} - \cancel{a_2^3 b_2} - \cancel{a_2^2 a_3 b_3} + \cancel{a_1^2 a_3 b_3} + \cancel{a_2^2 a_3 b_3} + a_3^3 b_3$$

$$= a_1 a_3^2 b_1 - a_2 a_3^2 b_2 - a_3^3 b_3$$

$$= \boxed{0}$$

SECTION 12.4

1 Find the cross product $\vec{a} \times \vec{b}$ and verify that it is orthogonal to both $\vec{a}$ and $\vec{b}$.

$\vec{a} = \langle 6, 0, -2 \rangle$, $\vec{b} = \langle 0, 8, 0 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 0 & -2 \\ 0 & 8 & 0 \end{vmatrix} = \hat{i}(16) - \hat{j}(0) + \hat{k}(48)$$
$$= 16\hat{i} + 48\hat{k} = \boxed{\langle 16, 0, 48 \rangle}$$

We check that:

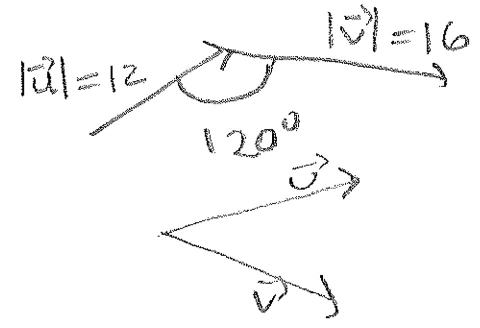
$$\vec{a} \cdot (\vec{a} \times \vec{b}) = \langle 6, 0, -2 \rangle \cdot \langle 16, 0, 48 \rangle = 96 - 96 = 0$$
$$\vec{b} \cdot (\vec{a} \times \vec{b}) = \langle 0, 8, 0 \rangle \cdot \langle 16, 0, 48 \rangle = 0$$

9 Find the vector, not with determinants, but by using properties of cross products.

$$(i \times j) \times k = k \times k = 0$$

(11) $$(j - k) \times (k - i) = (j - k) \times k - (j - k) \times i$$
$$= (j \times k) - \cancel{k \times k}^{0} - [(j \times i) - k \times i]$$
$$= i + k + j = \hat{i} + \hat{j} + \hat{k}$$

(15) Find $|\vec{u} \times \vec{v}|$ and determine whether $\vec{u} \times \vec{v}$ is directed into the page or out of the page.



$$|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin\theta$$
$$= 12 \cdot 16 \cdot \sin(120)$$
$$= 12 \cdot 16 \quad \frac{\sqrt{3}}{2}$$
$$\boxed{= 96\sqrt{3}}$$

Into the page.

(19) Find two unit vectors orthogonal to both $\vec{a}\langle 3,2,1\rangle$ and $\vec{b}\langle -1,1,0\rangle$

A vector orthogonal to both $\vec{a}$ and $\vec{b}$ is $\vec{a} \times \vec{b}$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \hat{i}(-1) - \hat{j}(1) + \hat{k}(3+2)$$
$$= -\hat{i} - \hat{j} + 5\hat{k}$$

We can check that indeed

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = \langle 3,2,1\rangle \cdot \langle -1,-1,5\rangle = -3-2+5 = \boxed{0}$$

Hence $\vec{a}$ is orthogonal to $\vec{a} \times \vec{b}$

AND,

$$\vec{b} \cdot (\vec{a} \times \vec{b}) = \langle -1,1,0\rangle \cdot \langle -1,-1,5\rangle = 1-1+0 = \boxed{0}$$

Hence, $\vec{b}$ is orthogonal to $\vec{a} \times \vec{b}$

To find a unit vector $\hat{u}$, we compute $\hat{u} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$$\Rightarrow \hat{u} = \frac{\langle -1,-1,5\rangle}{\sqrt{1+1+25}} = \frac{\langle -1,-1,5\rangle}{\sqrt{27}}$$

Hence, two unit vectors orthogonal to $\vec{a}$ and $\vec{b}$ are

$$\hat{u}_1 = \frac{1}{3\sqrt{3}}\langle -1,-1,5\rangle \quad \text{and} \quad \hat{u}_2 = \frac{-1}{3\sqrt{3}}\langle -1,-1,5\rangle$$

Pointing in opposite directions.

Section 12.4 :

(1) Find the cross product $\vec{a}\times\vec{b}$ and verify that it is orthogonal to both $\vec{a}$ and $\vec{b}$.

(1) $\vec{a} = \langle 6,0,-2\rangle$, $\vec{b} = \langle 0,8,0\rangle$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 0 & -2 \\ 0 & 8 & 0 \end{vmatrix} = \hat{i}(0+16) - \hat{j}(0) + \hat{k}(48) = \boxed{16\hat{i} + 48\hat{k}}$$

$\vec{a}\cdot(\vec{a}\times\vec{b}) = \langle 6,0,-2\rangle\cdot\langle 16,0,48\rangle = 96-96 = 0. \Rightarrow \vec{a}\perp\vec{a}\times\vec{b}$

$\vec{b}\cdot(\vec{a}\times\vec{b}) = \langle 0,8,0\rangle\cdot\langle 16,0,48\rangle = 0. \Rightarrow \vec{b}\perp\vec{a}\times\vec{b}$

(2) $\vec{a} = \langle 1,1,-1\rangle$, $\vec{b} = \langle 2,4,6\rangle$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 2 & 4 & 6 \end{vmatrix} = \hat{i}(6+4) - \hat{j}(6+2) + \hat{k}(4-2) = \boxed{10\hat{i} - 8\hat{j} + 2\hat{k}}$$

$\vec{a}\cdot(\vec{a}\times\vec{b}) = \langle 1,1,-1\rangle\cdot\langle 10,-8,2\rangle = 10-8-2 = 0 \Rightarrow \vec{a}\perp\vec{a}\times\vec{b}$

$\vec{b}\cdot(\vec{a}\times\vec{b}) = \langle 2,4,6\rangle\cdot\langle 10,-8,2\rangle = 20-32+12 = 0 \Rightarrow \vec{b}\perp\vec{a}\times\vec{b}$

(3) $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$, $\vec{b} = -\hat{i} + 5\hat{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 5 \end{vmatrix} = \hat{i}(15) - \hat{j}(5-2) + \hat{k}(3) = \boxed{15\hat{i} - 3\hat{j} + 3\hat{k}}$$

$\vec{a}\cdot(\vec{a}\times\vec{b}) = \langle 1,3,-2\rangle\cdot\langle 15,-3,3\rangle = 15-9-6 = 0 \Rightarrow \vec{a}\perp\vec{a}\times\vec{b}$

$\vec{b}\cdot(\vec{a}\times\vec{b}) = \langle -1,0,5\rangle\cdot\langle 15,-3,3\rangle = -15+15 = 0 \Rightarrow \vec{b}\perp\vec{a}\times\vec{b}$

(4) $\vec{a} = \hat{j} + 7\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 4\hat{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 7 \\ 2 & -1 & 4 \end{vmatrix} = \hat{i}(4+7) - \hat{j}(-14) + \hat{k}(-2) = \boxed{11\hat{i} + 14\hat{j} - 2\hat{k}}$$

$\vec{a}\cdot(\vec{a}\times\vec{b}) = \langle 0,1,7\rangle\cdot\langle 11,14,-2\rangle = 14-14 = 0 \Rightarrow \vec{a}\perp\vec{a}\times\vec{b}$

$\vec{b}\cdot(\vec{a}\times\vec{b}) = \langle 2,-1,4\rangle\cdot\langle 11,14,-2\rangle = 22-14-8 = 0 \Rightarrow \vec{b}\perp\vec{a}\times\vec{b}$

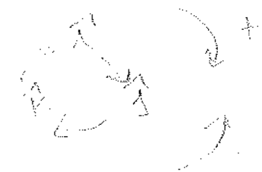
⑤ $\vec{a} = \hat{i} - \hat{j} - \hat{k}$, $\vec{b} = \frac{1}{2}\hat{i} + \hat{j} + \frac{1}{2}\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ 1/2 & 1 & 1/2 \end{vmatrix} = \hat{i}(-1/2+1) - \hat{j}\left(\frac{1}{2}+\frac{1}{2}\right) + \hat{k}\left(1+\frac{1}{2}\right) = \boxed{\frac{1}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k}}$$

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = \langle 1, -1, -1 \rangle \cdot \langle 1/2, -1, 3/2 \rangle = \frac{1}{2} + 1 - \frac{3}{2} = 0 \Rightarrow \vec{a} \perp \vec{a} \times \vec{b}$$

$$\vec{b} \cdot (\vec{a} \times \vec{b}) = \langle 1/2, 1, 1/2 \rangle \cdot \langle 1/2, -1, 3/2 \rangle = \frac{1}{4} - 1 + \frac{3}{4} = 0 \Rightarrow \vec{b} \perp \vec{a} \times \vec{b}$$

Now, using THEOREM 11:



$$\vec{a} \times \vec{b} = (\hat{i} - \hat{j} - \hat{k}) \times \left(\frac{1}{2}\hat{i} + \hat{j} + \frac{1}{2}\hat{k}\right)$$

$$= \hat{i} \times \hat{i} + (\hat{i} \times \hat{j}) + (\hat{i} \times \tfrac{1}{2}\hat{k})$$
$$-(\hat{j} \times \tfrac{1}{2}\hat{i}) - \cancel{(\hat{j} \times \hat{j})} - (\hat{j} \times \tfrac{1}{2}\hat{k})$$
$$-(\hat{k} \times \tfrac{1}{2}\hat{i}) - (\hat{k} \times \hat{j}) - (\hat{k} \times \tfrac{1}{2}\hat{k})$$

$$= (\hat{i} \times \hat{j}) + \frac{1}{2}(\hat{i} \times \hat{k}) - \frac{1}{2}(\hat{j} \times \hat{i}) - \frac{1}{2}(\hat{j} \times \hat{k}) - \frac{1}{2}(\hat{k} \times \hat{i}) - (\hat{k} \times \hat{j})$$

$$= \hat{k} - \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k} - \frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} + \hat{i} = \boxed{\frac{1}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k}}$$

---

6) $\vec{a} = t\hat{i} + \cos t\,\hat{j} + \sin t\,\hat{k}$; $\vec{b} = \hat{i} - \sin t\,\hat{j} + \cos t\,\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & \cos t & \sin t \\ 1 & -\sin t & \cos t \end{vmatrix} = \hat{i}(\cos^2 t + \sin^2 t) - \hat{j}(t\cos t - \sin t) + \hat{k}(-t\sin t - \cos t)$$

$$= \boxed{\hat{i} - (t\cos t - \sin t)\hat{j} - (t\sin t + \cos t)\hat{k}}$$

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = \langle t, \cos t, \sin t \rangle \cdot \langle 1, \sin t - t\cos t, -\cos t - t\sin t \rangle$$
$$= t + \cos t \sin t - t\cos^2 t - \cos t \sin t - t\sin^2 t$$
$$= t - t(\cos^2 t + \sin^2 t) = t - t(1) = 0 \Rightarrow \vec{a} \perp \vec{a} \times \vec{b}$$

$$\vec{b} \cdot (\vec{a} \times \vec{b}) = \langle 1, -\sin t, \cos t \rangle \cdot \langle 1, \sin t - t\cos t, -\cos t - t\sin t \rangle$$
$$= 1 - \sin^2 t + t\sin t\cos t - \cos^2 t - t\sin t\cos t$$
$$= 1 - (\sin^2 t + \cos^2 t) = 1 - 1 = 0 \Rightarrow \vec{b} \perp \vec{a} \times \vec{b}$$

(7) $\vec{a} = \langle t, 1, 1/t \rangle$, $\vec{b} = \langle t^2, t^2, 1 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ t & 1 & 1/t \\ t^2 & t^2 & 1 \end{vmatrix} = \hat{\imath}(1-t) - \hat{\jmath}\underbrace{(t-t)}^{0} + \hat{k}(t^3 - t^2)$$
$$= \boxed{(1-t)\hat{\imath} + (t^3-t^2)\hat{k}}$$

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = \langle t, 1, 1/t \rangle \cdot \langle 1-t, 0, t^3-t^2 \rangle$$
$$= \cancel{t} - \cancel{t^2} + \cancel{t^2} - \cancel{t} = 0 \implies \vec{a} \perp \vec{a} \times \vec{b}$$

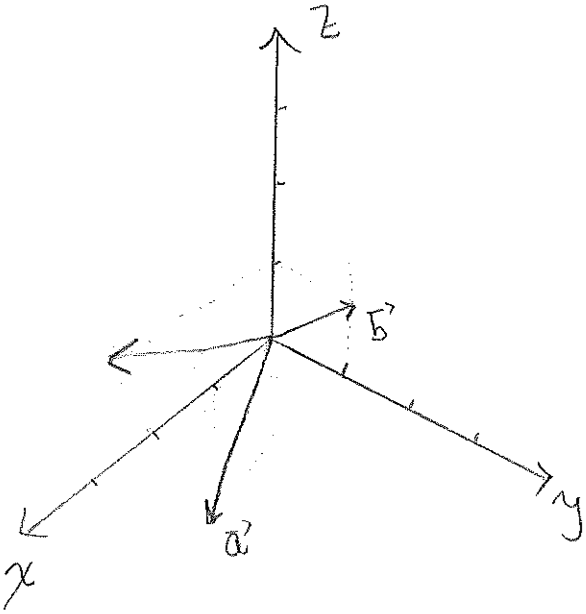
$$\vec{b} \cdot (\vec{a} \times \vec{b}) = \langle t^2, t^2, 1 \rangle \cdot \langle 1-t, 0, t^3-t^2 \rangle$$
$$= t^2 - t^3 + t^3 - t^2 = 0 \implies \vec{a} \perp \vec{a} \times \vec{b}$$

---

8) If $\vec{a} = \hat{\imath} - 2\hat{k}$ and $\vec{b} = \hat{\jmath} + \hat{k}$, find $\vec{a} \times \vec{b}$.
Sketch $\vec{a}, \vec{b}$ and $\vec{a} \times \vec{b}$ as vectors starting at the origin

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix} = \hat{\imath}(2) - \hat{\jmath}(1) + \hat{k}(1) = \boxed{2\hat{\imath} - \hat{\jmath} + \hat{k}}$$

$$\vec{a} \times \vec{b} = (\hat{\imath} - 2\hat{k}) \times (\hat{\jmath} + \hat{k}) = (\hat{\imath} \times \hat{\jmath}) + (\hat{\imath} \times \hat{k}) - 2(\hat{k} \times \hat{\jmath}) - 2(\hat{k} \times \hat{k})^{0}$$
$$= \hat{k} + (-\hat{\jmath}) + 2\hat{\imath} = \boxed{2\hat{\imath} - \hat{\jmath} + \hat{k}}$$



9) Find the vector, not with determinants, but by using properties of C.P:

$(\hat{i}\times\hat{j})\times\hat{k} = \hat{k}\times\hat{k} = \boxed{\vec{0}}$

10) $\vec{k}\times(\hat{i}-2\hat{j}) = (\hat{k}\times\hat{i}) - 2(\hat{k}\times\hat{j}) = \hat{j}+2\hat{i} = \boxed{2\hat{i}+\hat{j}}$

$(2\hat{i}+\hat{j})\cdot\hat{k} = \langle 2,0,0\rangle\cdot\langle 0,0,1\rangle = 0 \Rightarrow (2\hat{i}+\hat{j}) \perp \hat{k}$

$(\hat{i}-2\hat{j})\cdot(2\hat{i}+\hat{j}) = \langle 1,-2,0\rangle\cdot\langle 2,1,0\rangle = 2-2 = 0 \Rightarrow (\hat{i}-2\hat{j}) \perp (2\hat{i}+\hat{j})$

11) $(\hat{j}-\hat{k})\times(\hat{k}-\hat{i}) = (\hat{j}\times k) - (\hat{j}\times\hat{i}) - \cancel{(\hat{k}\times\hat{k})} + (\hat{k}+\hat{i})$

$= \hat{i}+\hat{k}+\hat{j} = \boxed{\hat{i}+\hat{j}+\hat{k}}$

$(\hat{j}-\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k}) = \langle 0,1,-1\rangle\cdot\langle 1,1,1\rangle = 0+1-1 = 0$

$(\hat{k}-\hat{i})\cdot(\hat{i}+\hat{j}+\hat{k}) = \langle -1,0,1\rangle\cdot\langle 1,1,1\rangle = -1+0+1 = 0$

12) $(\hat{i}+\hat{j})\times(\hat{i}-\hat{j}) = \underbrace{\cancel{\hat{i}\times\hat{i}}}_{0} - (\hat{i}\times\hat{j}) + (\hat{j}\times\hat{i}) \underbrace{\cancel{-\hat{j}\times\hat{j}}}_{0}$

$= -\hat{k} - \hat{k} = -2\hat{k}$

$(\hat{i}+\hat{j})\cdot(-2\hat{k}) = \langle 1,1,0\rangle\cdot\langle 0,0,-2\rangle = 0 \Rightarrow (\hat{i}+\hat{j}) \perp -2\hat{k}$

$(\hat{i}-\hat{j})\cdot(-2\hat{k}) = \langle 1,-1,0\rangle\cdot\langle 0,0,-2\rangle = 0 \Rightarrow (\hat{i}-\hat{j}) \perp -2\hat{k}$

13 State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.

(a) $\vec{a}\cdot(\vec{b}\times\vec{c})$ vector · vector = $\boxed{\text{number}}$ ✓

(b) $\vec{a}\times(\vec{b}\cdot\vec{c})$ vector × number = $\boxed{\text{Not meaningful}}$ ✓

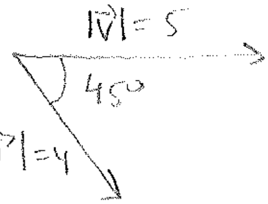
(c) $\vec{a}\times(\vec{b}\times\vec{c})$ vector × (vector × vector) = vector × vector = $\boxed{\text{vector}}$ ✓

(d) $\vec{a}\cdot(\vec{b}\cdot\vec{c})$ vector · (vector · vector) = vector · number = $\boxed{\text{Not meaningful}}$

(e) $(\vec{a}\cdot\vec{b})\times(\vec{c}\cdot\vec{d})$ (vec · vec) × (vec · vec) = number × number = $\boxed{\text{Not meaningful}}$ ✓

(f) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$ (vec × vec) · (vec × vec) = vector · vector = $\boxed{\text{number}}$ ✓

(14)

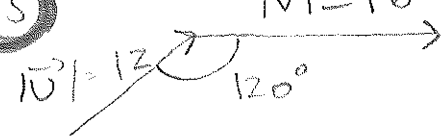


Direction of $\vec{u}\times\vec{v}$ is out of the page

$|\vec{u}\times\vec{v}| = |\vec{u}|\cdot|\vec{v}|\cdot\sin\theta$

$= 5\cdot 4\cdot\sin(45°) = 20\,\frac{\sqrt{2}}{2} = 10\sqrt{2}$

15



$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\cdot\sin\theta$

$= 12\cdot 16\cdot\sin 60°$

$= 12\times \overset{8}{16}\times\frac{\sqrt{3}}{2} = 96\sqrt{3}$

Direction of $\vec{u}\times\vec{v}$ into the page

18) If $\vec{a} = \langle 1,0,1 \rangle$, $\vec{b} = \langle 2,1,-1 \rangle$, and $\vec{c} = \langle 0,1,3 \rangle$, show that

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \hat{\imath}(-1) - \hat{\jmath}(-1-2) + \hat{k}(1) = \boxed{-\hat{\imath} + 3\hat{\jmath} + \hat{k}}$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -1 \\ 0 & 1 & 3 \end{vmatrix} = \hat{\imath}(3+1) - \hat{\jmath}(6) + \hat{k}(2) = \boxed{4\hat{\imath} - 6\hat{\jmath} + 2\hat{k}}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 1 \\ 4 & -6 & 2 \end{vmatrix} = \hat{\imath}(6) - \hat{\jmath}(-2) + \hat{k}(-6) = \boxed{6\hat{\imath} + 2\hat{\jmath} - 6\hat{k}}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} = \hat{\imath}(8) - \hat{\jmath}(-3) + \hat{k}(-1) = \boxed{8\hat{\imath} + 3\hat{\jmath} - \hat{k}}$$

Hence, $6\hat{\imath} + 2\hat{\jmath} - 6\hat{k} \neq 8\hat{\imath} + 3\hat{\jmath} - \hat{k}$

since $|\vec{a} \times (\vec{b} \times \vec{c})| = \sqrt{36+4+36} = \sqrt{76} \neq$

$|(\vec{a} \times \vec{b}) \times \vec{c}| = \sqrt{64+9+1} = \sqrt{74}$

19) Find two unit vectors orthogonal to both $\langle 3,2,1 \rangle$ and $\langle -1,1,0 \rangle$

$\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$; $\vec{b} = -\hat{\imath} + \hat{\jmath}$

$$\vec{a} \times \vec{b} = (3\hat{\imath} + 2\hat{\jmath} + \hat{k}) \times (-\hat{\imath} + \hat{\jmath})$$

$$= -3\cancel{(\hat{\imath} \times \hat{\imath})} + 3(\hat{\imath} \times \hat{\jmath}) - 2(\hat{\jmath} \times \hat{\imath}) + 2\cancel{(\hat{\jmath} \times \hat{\jmath})} - (\hat{k} \times \hat{\imath}) + (\hat{k} \times \hat{\jmath})$$

$$= 3\hat{k} + 2\hat{k} - \hat{\jmath} - \hat{\imath} = \boxed{-\hat{\imath} - \hat{\jmath} + 5\hat{k}} = \langle -1,-1,5 \rangle$$

Unit vector: $\hat{u} = \dfrac{\langle -1,-1,5 \rangle}{\sqrt{1+1+25}} = \langle -1/3\sqrt{3}, -1/3\sqrt{3}, 5/3\sqrt{3} \rangle$

Two unit vectors are: $\hat{u}_1 = \langle -1/3\sqrt{3}, -1/3\sqrt{3}, 5/3\sqrt{3} \rangle$ AND

$\hat{u}_2 = \langle 1/3\sqrt{3}, 1/3\sqrt{3}, -5/3\sqrt{3} \rangle$

Review Rules:

(1.) $a \times b = -b \times a$

(2.) $(ca) \times b = c(a \times b) = a \times c(b)$

(3) $a \times (b+c) = a \times b + a \times c$

(4) $(a+b) \times c = a \times c + b \times c$

Section 12.5

(2) Find a vector equation and parametric equations for the line through the point $(6,-5,2)$ and parallel to the vector $\langle 1, 3, -2/3 \rangle$.

$$\vec{r}(t) = \vec{r_0} + t\vec{v} = (6,-5,2) + t\langle 1,3,-2/3\rangle$$
$$= \boxed{\langle 6+t, -5+t, 2-\tfrac{2}{3}t\rangle = \vec{r}(t)}$$

(3) through the point $(2, 2.4, 3.5)$ and parallel to the vector $3\hat{i} + 2\hat{j} - \hat{k}$.

$$\vec{r}(t) = \vec{r_0} + t\vec{v} = (2, 2.4, 3.5) + t\langle 3, 2, -1\rangle$$
$$= \boxed{\langle 2+3t, 2.4+2t, 3.5-t\rangle}$$
$$= \boxed{(2\hat{i} + 2.4\hat{j} + 3.5\hat{k}) + t(3\hat{i} + 2\hat{j} - \hat{k})}$$

(4) through the point $(0,14,-10)$ and parallel to the line $x = -1+2t$, $y = 6-3t$, $z = 3+9t$

$$\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$$
$$\Rightarrow x = x_0 + ta \; ; \; y = y_0 + tb \; ; \; z = z_0 + tc$$

$$\vec{r}(t) = (0,14,-10) + t\langle 2,-3,9\rangle = \boxed{\langle 2t, 14-3t, -10+9t\rangle}$$

5) the line through the point (1, 0, 6) and perpendicular to the plane $x + 3y + z = 5$

$\vec{v} = \langle 1, 3, 1 \rangle \Rightarrow \vec{r}(t) = (1, 0, 6) + t\langle 1, 3, 1 \rangle$

the vector equation $\boxed{= (\hat{\imath} + 6\hat{k}) + t(\hat{\imath} + 3\hat{\jmath} + \hat{k})}$

the parametric equations: $\boxed{x(t) = 1 + t \; ; \; y(t) = 3t \; ; \; z(t) = 6 + t}$

6) the line through the origin and the point (4, 3, -1)

We first need to compute the direction of this line.

the direction vector is: $\vec{v} = (4, 3, -1) - (0, 0, 0) = \langle 4, 3, -1 \rangle$

Vector equation:

$$\vec{r}(t) = (0, 0, 0) + t\langle 4, 3, -1 \rangle = t\langle 4, 3, -1 \rangle$$
$$= 4t\,\hat{\imath} + 3t\,\hat{\jmath} - t\,\hat{k}$$
$$\boxed{= t(4\hat{\imath} + 3\hat{\jmath} - \hat{k})}$$

Parametric equation:

$x(t) = 4t \; ; \; y(t) = 3t \; ; \; z(t) = -t$

Symmetric equations:

$$\frac{x}{4} = \frac{y}{3} = \frac{z}{-1}$$

7) the line through the points $(0, \frac{1}{2}, 1)$ and $(2, 1, -3)$.

the direction vector is given by: $\vec{v} = (2, 1, -3) - (0, \frac{1}{2}, 1)$

$\vec{v} = \langle 2, 1/2, -4 \rangle$

the vector equation: $\vec{r}(t) = (0, \frac{1}{2}, 1) + t\langle 2, 1/2, -4 \rangle = \langle 2t, \frac{1}{2} + \frac{1}{2}t, 1 - 4t \rangle$

$$\boxed{\vec{r}(t) = (\tfrac{1}{2}\hat{\jmath} + \hat{k}) + t(2\hat{\imath} + 1/2\,\hat{\jmath} - 4\hat{k})}$$

Parametric equations

$x(t) = 2t \; ; \; y(t) = \frac{1}{2} + \frac{1}{2}t \; ; \; z(t) = 1 - 4t$

Symmetric Equations:

$x = y - 1/2 = z - 1$

(9) the line through the points $(-8,1,4)$ and $(3,-2,4)$.

the direction $\vec{v}$ is given by : $\vec{v} = \langle 3,-2,4\rangle - \langle -8,1,4\rangle$

$$\boxed{\vec{v} = \langle 11,-3,0\rangle}$$

Vector equation:

$$\vec{r}(t) = \vec{r_0} + t\,\vec{v}$$

In this case,

$$\vec{r}(t) = (3,-2,4) + t\langle 11,-3,0\rangle = \langle 3+11t, -2-3t, 4\rangle$$

$$= (3\hat{i} - 2\hat{j} + 4\hat{k}) + t(11\hat{i} - 3\hat{j})$$

Parametric equations:

$$x(t) = 3 + 11t;\quad y(t) = -2-3t,\quad z(t) = 4$$

Symmetric equations:

$$\frac{x-3}{11} = \frac{y+2}{-3};\quad z = 4$$

(13) Is the line through $(-4,-6,1)$ and $(-2,0,-3)$ parallel to the line through $(10,18,4)$ and $(5,3,14)$?

We need to compte the direction vectors for each line.

For L_1: $\vec{v_1} = (-4,-6,1) - (-2,0,-3) = \langle -2,-6,4\rangle$

For L_2: $\vec{v_2} = (10,18,4) - (5,3,14) = \langle 5,15,-10\rangle$

Now, if we take the cross product :

$$\vec{v_1} \times \vec{v_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -6 & 4 \\ 5 & 15 & -10 \end{vmatrix} = \hat{i}(60-60) - \hat{j}(20-20) + \hat{k}(-30+30)$$

$$= \vec{0}$$

$\Rightarrow$ the vectors (lines) are parallel

Another way to see this is: $\vec{v_2} = -\frac{5}{\;}\vec{v_1} \Rightarrow \vec{v_1}$ is parallel

(15) (a) Find symmetric equations for the line that passes through the point $(1,-5,6)$ and is parallel to the vector $\langle -1,2,-3\rangle$.

$$\frac{x-1}{-1} = \frac{y+5}{2} = \frac{z-6}{-3}$$

(b) Find the points in which the required line in part (a) intersects the coordinate planes.

(i) the point of intersection with the xy-plane is of the form : $(x_0, y_0, 0)$.
Plugging this into our equations:

•) $\frac{x_0-1}{-1} = \frac{+6}{3} \Rightarrow x_0 - 1 = -2 \Rightarrow \boxed{x_0 = -1}$

•) $\frac{y_0+5}{2} = \frac{+6}{3} \Rightarrow y_0 + 5 = 4 \Rightarrow \boxed{y_0 = -1}$

the intersection with the $\boxed{\text{xy-plane is } (-1,-1,0)}$

(ii) the point of intersection with the xz-plane is of the form $(x_0, 0, z_0)$. then

•) $\frac{x_0-1}{-1} = \frac{5}{2} \Rightarrow x_0 = -\frac{5}{2} + 1 \Rightarrow \boxed{x_0 = -\frac{3}{2}}$

•) $\frac{z_0-6}{-3} = \frac{5}{2} \Rightarrow z_0 = -\frac{15}{2} + 6 \Rightarrow \boxed{z_0 = -\frac{3}{2}}$

the intersection with the $\boxed{\text{xz-plane is } (-3/2, 0, -3/2)}$

(iii) the point of intersection with the yz-plane is of the form $(0, y_0, z_0)$.

•) $\frac{y_0+5}{2} = 1 \Rightarrow y_0 + 5 = 2 \Rightarrow \boxed{y_0 = -3}$

•) $\frac{z_0-6}{-3} = 1 \Rightarrow z_0 - 6 = -3 \Rightarrow \boxed{z_0 = 3}$

$\boxed{(0,-3,3)}$

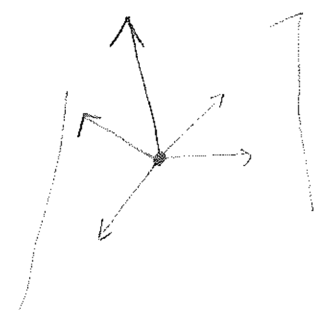
23) Find an equation of the plane through the origin and perpendicular to the vector $\langle 1,-2,5\rangle$.

$$P := \vec{n}\cdot\langle x,y,z\rangle = 0$$

$$\langle 1,-2,5\rangle\cdot\langle x,y,z\rangle = 0$$

$$\boxed{x-2y+5z=0}$$

25) the plane through the point $(-1,\tfrac{1}{2},3)$ and with normal vector $\hat{i}+4\hat{j}+\hat{k}$.



$$P := \vec{n}\cdot(\vec{r}-\vec{r}_0) = 0$$

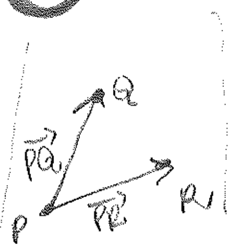
$$\langle 1,4,1\rangle\cdot\left(\langle x,y,z\rangle-\langle -1,\tfrac{1}{2},3\rangle\right)=0$$

$$(x+1)+4(y-\tfrac{1}{2})+(z-3)=0$$

$$x+4y+z+1-2-3=0$$

$$\boxed{x+4y+z=4}$$

31) the plane through the points $(0,1,1)$, $(1,0,1)$ and $(1,1,0)$



Let $P(0,1,1)$, $Q(1,0,1)$ and $R(1,1,0)$.

the vectors $\vec{PQ}$ and $\vec{QR}$ are contained in the plane.

$$\vec{PQ} = (1,0,1)-(0,1,1) = \langle 1,-1,0\rangle = \hat{i}-\hat{j}$$

$$\vec{QR} = (1,1,0)-(1,0,1) = \langle 0,1,-1\rangle = \hat{j}-\hat{k}$$

the vector $\vec{PQ}\times\vec{QR}$ is perpendicular to both $\vec{PQ}$ and $\vec{QR}$ and thus, is the normal vector of the desired plane.

$$\vec{PQ}\times\vec{QR} = (\hat{i}-\hat{j})\times(\hat{j}-\hat{k}) = (\hat{i}\times\hat{j})-(\hat{i}\times\hat{k})-\cancel{(\hat{j}\times\hat{j})}^{0}+(\hat{j}\times\hat{k})$$

$$= \hat{k}+\hat{j}+\hat{i} = \boxed{\hat{i}+\hat{j}+\hat{k}} = \vec{n}$$

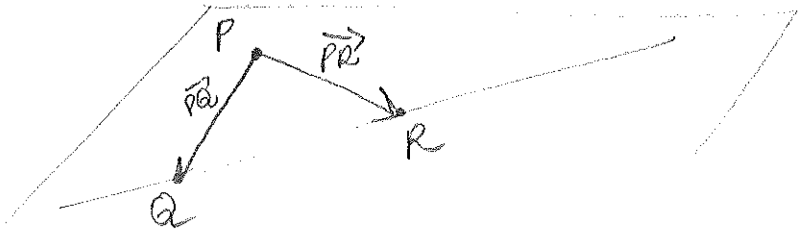
So our plane is:

$$\vec{n}\cdot(\vec{r}-\vec{r}_0)=0 \iff \langle 1,1,1\rangle\cdot\langle x-0,y-1,z-1\rangle=0$$

$$\iff x+y-1+z-1=0 \iff \boxed{x+y+z=2}$$

We can check that this eq. contains our three initial points.

(35) the plane that passes through the point $(6,0,-2)$ and contains the line $x=4-2t$, $y=3+5t$, $z=7+4t$.



Let $P(6,0,-2)$; and Q, R be two points on the line.

If $t=0 \Rightarrow Q(4,3,7)$

If $t=1 \Rightarrow R(2,8,11)$

Now we can compute two vectors on the plane:

$$\vec{PQ} = (4,3,7)-(6,0,-2) = \langle -2,3,9 \rangle$$

$$\vec{PR} = (2,8,11)-(6,0,-2) = \langle -4,8,13 \rangle$$

the normal vector $\vec{n}$ to the plane is $\vec{PQ} \times \vec{PR}$, since $\vec{n}$ is perpendicular to both of these.

$$\vec{PQ} \times \vec{PQ} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 3 & 9 \\ -4 & 8 & 13 \end{vmatrix} = \hat{\imath}(39-72) - \hat{\jmath}(-26+36) + \hat{k}(-16+12)$$

$$= \boxed{-33\hat{\imath} - 10\hat{\jmath} - 4\hat{k}} = \vec{n}$$

Finally, the plane is given by:

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \iff \vec{n} \cdot \langle x-6, y, z+2 \rangle = 0$$

$$\langle -33,-10,-4 \rangle \cdot \langle x-6, y, z+2 \rangle = 0$$

$$-33x + 198 - 10y - 4z - 8 = 0$$

$$-33x - 10y - 4z = -190$$

Multiply both sides by -1

$$\boxed{33x + 10y + 4z = 190}$$

Finally, check that all points

45 Find the points at which the line intersects the given plane.

$x = 3-t$, $y = 2+t$, $z = 5t$; $x - y + 2z = 9$.

Replace the parametric eqs. for the line into the plane to obtain t:

$$3-t-2-t+10t = 9 \Rightarrow 8t = 8 \Rightarrow t = 1$$

Hence, the point is

$x = 3-1 = 2$; $y = 2+1 = 3$; $z = 5(1) = 5$, i.e.,

$(2,3,5)$. this point is both in the line and the plane.

61 Find an equation for the plane consisting of all points that are equidistant from the points $(1,0,-2)$ and $(3,4,0)$

$$d((x,y,z),(1,0,-2)) = d((x,y,z),(3,4,0))$$

$$\sqrt{(x-1)^2 + y^2 + (z+2)^2} = \sqrt{(x-3)^2 + (y-4)^2 + z^2}$$

$$(x-1)^2 + y^2 + (z+2)^2 = (x-3)^2 + (y-4)^2 + z^2$$

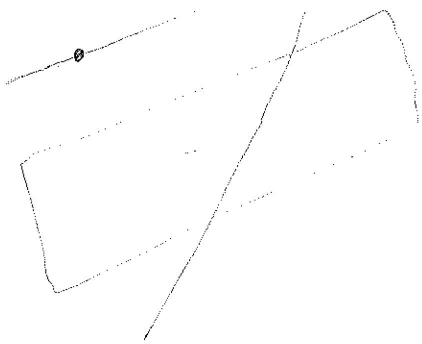
$$\cancel{x^2} - 2x + 1 + \cancel{y^2} + \cancel{z^2} + 4z + 4 = \cancel{x^2} - 6x + 9 + \cancel{y^2} - 8y + 16 + \cancel{z^2}$$

$$-2x + 6x + 8y + 4z + 1 + 4 - 9 - 16 = 0$$

$$\boxed{4x + 8y + 4z = 20} \Leftrightarrow 2x + 4y + 2z = 10$$

$$\Leftrightarrow \boxed{x + 2y + z = 5}$$

05 Find parametric equations for the line through the point $(0,1,2)$ that is parallel to the plane $x+y+z=2$ and perpendicular to the line $x=1+t$, $y=1-t$, $z=2t$.



The parallel line through the point has the vector equation:

$$\vec{r}(t) = \langle 0,1,2\rangle + t\langle a,b,c\rangle,$$

where we have to find a, b, c.

(i) Since the line is parallel to the plane $x+y+z=2$, it has to be perpendicular to the normal vector $\langle 1,1,1\rangle$

$$\langle a,b,c\rangle \cdot \langle 1,1,1\rangle = 0$$

$$\Rightarrow \quad a+b+c=0$$

(ii) Since the line is perpendicular to the other line, it is perpendicular to the director vector $\langle 1,-1,2\rangle$

$$\langle a,b,c\rangle \cdot \langle 1,-1,2\rangle = 0$$

$$\Rightarrow \quad a-b+2c=0$$

Hence, we can solve the system

$$\begin{aligned} a+b+c=0 &\Rightarrow a=-b-c \\ a-b+2c=0 & \qquad \Downarrow \\ & -b-c-b+2c=0 \\ & -2b+c=0 \Rightarrow c=2b \end{aligned}$$

Choose a value for b, say $b=1 \Rightarrow c=2 \Rightarrow a=-3$

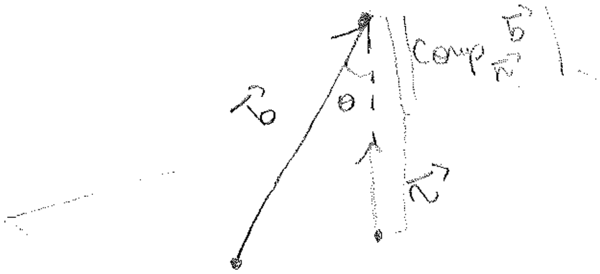
Hence, the desired line is

$$\vec{r}(t) = \langle 0,1,2\rangle + t\langle -3,1,2\rangle$$

$$\Leftrightarrow \quad x(t)=-3t \; ; \; y(t)=1+t \; ; \; z(t)=2+2t$$

71) Find the distance from the point to the given plane:

$(1,-2,4)$; $3x+2y+6z=5$



$\vec{n}\cdot\vec{b} = |\vec{n}||\vec{b}|\cdot\cos\theta$

From the picture

$\text{Comp}_{\vec{n}}\vec{b} = |b|\cdot\cos\theta$

$\Rightarrow \vec{n}\cdot\vec{b} = |\vec{n}|\cdot\text{Comp}_{\vec{n}}\vec{b}$

$\Rightarrow \text{Comp}_{\vec{n}}\vec{b} = \dfrac{\vec{n}\cdot\vec{b}}{|\vec{n}|}$

$$\Rightarrow \left|\text{Comp}_{\vec{n}}\vec{b}\right| = \frac{|\vec{n}\cdot\vec{b}|}{|\vec{n}|}$$

In this case: Pick a point on the plane:
for instance, $(1,1,0)$. Construct $\vec{b}$

$$\vec{b} = (1,-2,4)-(1,1,0) = \langle 0,-3,4\rangle$$

Now, the distance is given by

$$\left|\text{Comp}_{\vec{n}}\vec{b}\right| = \frac{|\langle 3,2,6\rangle\cdot\langle 0,-3,4\rangle|}{\sqrt{9+4+36}} = \frac{|-6+24|}{\sqrt{49}} = \boxed{\frac{18}{7}}$$

__Note__. __Interestingly__, if we pick __ANY__ other point on the plane, the answer remains the same

Pick: $(0,-\frac{1}{2},1)$. then $\vec{b} = (1,-2,4)-(0,-\frac{1}{2},1) = \langle 1,-\frac{3}{2},3\rangle$

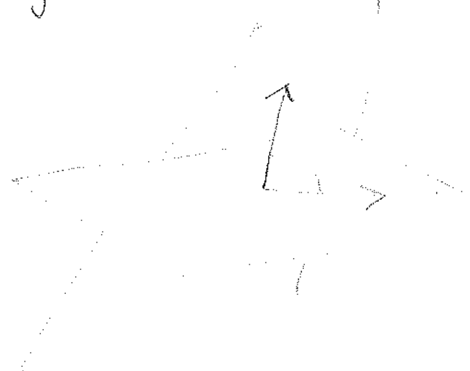
Compute the distance:

$$\left|\text{Comp}_{\vec{n}}\vec{b}\right| = \frac{|\vec{b}\cdot\vec{n}|}{|\vec{n}|} = \frac{|\langle 1,-\frac{3}{2},3\rangle\cdot\langle 3,2,6\rangle|}{7} = \frac{|3-3+18|}{7} = \boxed{\frac{18}{7}}$$

SECTION 12.5

(51) Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.

$x + 4y - 3z = 1$; $-3x + 6y + 7z = 0$.



TAKE the two normal vectors to the plane:

$\vec{n_1} = \langle 1, 4, -3 \rangle$

$\vec{n_2} = \langle -3, 6, 7 \rangle$

Compute the dot product: $\vec{n_1} \cdot \vec{n_2} = \langle 1, 4, -3 \rangle \cdot \langle -3, 6, 7 \rangle$
$= -3 + 24 - 21 = 0$

Hence, the two planes are $\boxed{\text{perpendicular}}$

(53) $x + y + z = 1$; $x - y + z = 1$.

$\vec{n_1} = \langle 1, 1, 1 \rangle$; $\vec{n_2} = \langle 1, -1, 1 \rangle$

$\vec{n_1} \cdot \vec{n_2} = \langle 1, 1, 1 \rangle \cdot \langle 1, -1, 1 \rangle = 1 - 1 + 1 = 1 \Rightarrow$ the planes are not perpendicular. the planes are also not parallel since there exists no $\alpha \in \mathbb{R}$, such that $\vec{n_1} = \alpha \vec{n_2}$.

therefore, there is an angle between the planes. To compute the angle between the plane is equivalent to compute the angle between normal vectors:

$\vec{n_1} \cdot \vec{n_2} = 1$

$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}| \cdot |\vec{n_2}| \cdot \cos\theta$
$= 3 \cdot \cos\theta$

$\Rightarrow \arccos\left(\frac{1}{3}\right) = \theta \Rightarrow \boxed{\theta \approx 70.5^\circ}$

55 $X = 4y - 2z$; $8y = 1 + 2x + 4z$

$\vec{n_1} = \langle 1, -4, 2 \rangle$ $\vec{n_2} = \langle 2, -8, 4 \rangle$

Since, $2 \cdot \vec{n_1} = 2 \cdot \langle 1, -4, 2 \rangle = \langle 2, -8, 4 \rangle = \vec{n_2}$

these planes are $\boxed{\text{parallel}}$ ✓

67 $x + y + z = 1$, $x + 2y + 2z = 1$

(a) Find parametric equations for the line of intersection of the planes.

$$\begin{cases} x + y + z = 1 \Rightarrow x = 1 - y - z \\ x + 2y + 2z = 1 \end{cases}$$

$$1 - y - z + 2y + 2z = 1$$

$$\Rightarrow y + z = 0 \Rightarrow y = -z$$

Let $z = t$. then: $y = -t$ and $x = 1 + t - t = 1$

the parametric equations are:

$\boxed{x(t) = 1 \ ; \ y(t) = -t \ ; \ z(t) = t}$ ✓

direction vector: $\langle 0, -1, 1 \rangle$

point on the line: $\langle 1, 0, 0 \rangle$

(b) Find the angle between the planes.

$\vec{n_1} = \langle 1, 1, 1 \rangle$; $\vec{n_2} = \langle 1, 2, 2 \rangle$

$$\vec{n_1} \cdot \vec{n_2} = \langle 1, 1, 1 \rangle \cdot \langle 1, 2, 2 \rangle = 1 + 2 + 2 = 5$$

$$\vec{n_1} \cdot \vec{n_2} = \sqrt{3}\sqrt{9} \cdot \cos\theta = 3\sqrt{3}\cos\theta$$

$$\Rightarrow \arccos\left(\frac{5}{3\sqrt{3}}\right) = \theta \Rightarrow \boxed{\theta \approx 15.7^\circ}$$ ✓

[Sectio]n 13.1:

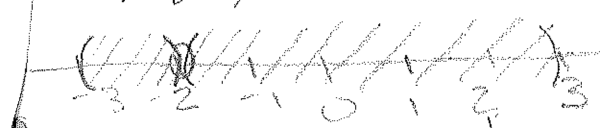
[Fi]nd the domain of the vector function

$\vec{r}(t) = \frac{t-2}{t+2}\hat{\imath} + \sin t\,\hat{\jmath} + \ln(9-t^2)\hat{k}$

Domain:
$(-3,-2) \cup (-2,3)$

$t+2 \neq 0 \Rightarrow t \neq -2$ $\Rightarrow -3 < t < 3$

$9-t^2 > 0 \Rightarrow 9 > t^2 \Rightarrow |t| < 3$



Here, the domain is $\boxed{t \in (-3,3) \setminus \{-2\}}$

[2)] Find the limit

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$

$\lim_{t\to 0}\left(e^{-3t}\hat{\imath} + \frac{t^2}{\sin^2 t}\hat{\jmath} + \cos(2t)\hat{k}\right) =$

$\lim_{t\to 0}\left\langle e^{-3t}, \frac{t^2}{\sin^2 t}, \cos(2t)\right\rangle =$

$\left\langle \lim_{t\to 0} e^{-3t}, \lim_{t\to 0}\frac{t^2}{\sin^2 t}, \lim_{t\to 0}\cos(2t)\right\rangle =$

$\left\langle e^{-3\cdot 0}, \lim_{t\to 0}\frac{\cancel{2}t}{\cancel{2}\cos(t)\sin(t)}, \cos(2\cdot 0)\right\rangle =$

$\left\langle 1, \lim_{t\to 0}\frac{1}{-\sin^2(t)+\cos^2(t)}, 1\right\rangle = \langle 1, 1, 1 \rangle$

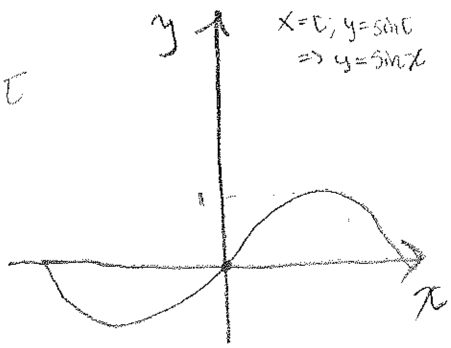
$= \boxed{\hat{\imath} + \hat{\jmath} + \hat{k}}$

[3)] $\vec{r}(t) = \langle t, \sin t, 2\cos t \rangle$

$x(t) = t$
$y(t) = \sin t$
$z(t) = 2\cos t$

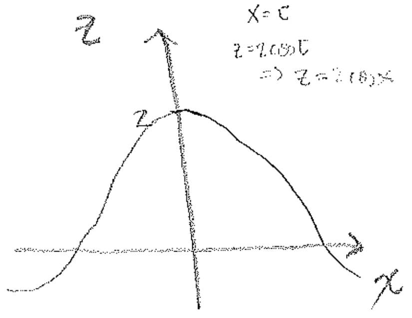
xy-plane

$x = t;\ y = \sin t$
$\Rightarrow y = \sin x$



xz-plane

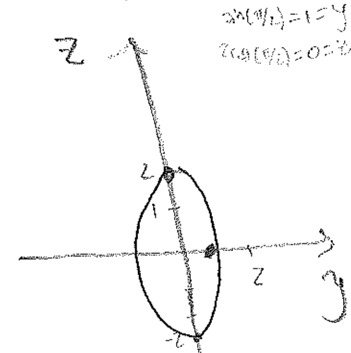
$x = t$
$z = 2\cos t$
$\Rightarrow z = 2\cos x$



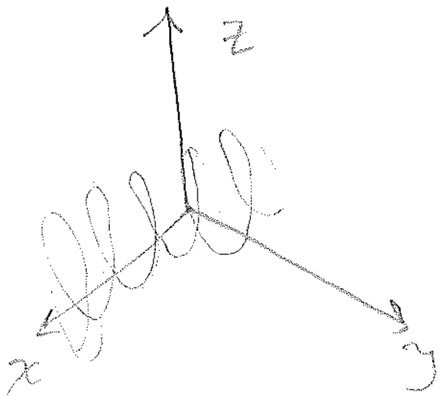
yz-plane

$t=0$
$\Rightarrow \sin 0 = 0 = y$
$2\cos 0 = 2\cdot 1 = 2 = z$

$t = \pi/2$
$\sin(\pi/2) = 1 = y$
$2\cos(\pi/2) = 0 = z$



$t = \pi \Rightarrow \sin(\pi) = 0 = y$
$2\cos(\pi) = 2\cdot(-1) = -2 = z$



19) Find a vector equation and parametric equations for the line segment that joins P to Q.

$P(0,-1,1)$, $Q(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$

The direction is given by $\vec{PQ} = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}) - (0,-1,1)$
$= \langle \frac{1}{2}, \frac{4}{3}, -\frac{3}{4} \rangle$

the vector equation is:

$$\boxed{\begin{aligned} \vec{r}(t) &= \langle 0,-1,1\rangle + t\langle \tfrac{1}{2}, \tfrac{4}{3}, -\tfrac{3}{4}\rangle \\ \vec{r}(t) &= (-\hat{j} + \hat{k}) + t(\tfrac{1}{2}\hat{i} + \tfrac{4}{3}\hat{j} - \tfrac{3}{4}\hat{k}) \end{aligned}}$$

the parametric equations are:

$$\boxed{x(t) = \frac{1}{2}t \;;\; y(t) = \frac{4}{3}t - 1 \;,\; z(t) = 1 - \frac{3}{4}t}$$

29) At what points does the curve $\vec{r}(t) = t\hat{i} + (2t - t^2)\hat{k}$ intersects the paraboloid $z = x^2 + y^2$?

$\vec{r}(t) = t\hat{i} + (2t - t^2)\hat{k} \Rightarrow x(t) = t \;;\; y(t) = 0 \;;$
$z(t) = 2t - t^2.$

Replace these in the eq. for the paraboloid.

$z = x^2 + y^2 \iff 2t - t^2 = t^2 + 0^2 \iff 2t^2 - 2t = 0 \iff t^2 - t = 0$

$\iff t(t-1) = 0 \Rightarrow \boxed{t = 0}$ OR $\boxed{t = 1}$

Hence, the points of intersection are:

$\boxed{\vec{r}(0) = \langle 0,0,0 \rangle}$ AND $\boxed{\vec{r}(1) = \langle 1,0,1 \rangle}$

SIMILAR TO 30

80) At what point does the helix $\vec{r}(t) = \langle \sin t, \cos t, t \rangle$ intersect the sphere $x^2+y^2+z^2=17$?

From the helix vector eq, we can obtain parametric eqs:

$$\vec{r}(t) = \langle \sin t, \cos t, t \rangle \iff x(t)=\sin t;\ y(t)=\cos t;\ z(t)=t$$

We can replace these eqs in the eq for the sphere:

$x^2+y^2+z^2=17$ intersects the helix when

$$(\sin t)^2+(\cos t)^2+t^2= \iff (\sin^2 t+\cos^2 t)+t^2=17$$
$$\iff t^2+1=17$$
$$\iff t^2=16 \Rightarrow \boxed{t=4} \text{ or } \boxed{t=-4}$$

Hence, when $t=4$ or $t=-4$ the helix intersects the sphere.
To get the points just plug values of t on the eq of the helix:

$$\vec{r}(4) = \langle \sin(4), \cos(4), 4 \rangle \Rightarrow (\sin(4), \cos(4), 4)$$
$$\vec{r}(-4) = \langle \sin(-4), \cos(-4), -4 \rangle \Rightarrow (\sin(-4), \cos(-4), -4)$$

39) SHOW THAT THE CURVE WITH PARAMETRIC EQ

$$x(t)=t^2,\quad y=1-3t;\quad z=1+t^3$$

PASSES THROUGH THE POINTS $(1,4,0)$ and $(9,-8,28)$ but NOT THROUGH THE POINT $(4,7,-6)$

i) For $(1,4,0)$

$$\begin{cases} x=t^2=1 \Rightarrow t^2=1 \\ y=1-3t=4 \Rightarrow 1-3t=4 \Rightarrow -3=3t \Rightarrow \boxed{t=-1} \\ z=1+t^3=0 \Rightarrow t^3=-1 \end{cases}$$

Since $t=-1$ satisfy all three equations

We get that $(x(-1), y(-1), z(-1)) = (1,4,0)$, which shows THAT THE CURVE PASSES THROUGH THIS point.

ii) For $(9,-8,28)$

$$\begin{cases} x=t^2=9 \Rightarrow \boxed{t=3} \text{ or } t=-3 \\ y=1-3t=-8 \\ z=1+t^3=28 \end{cases}$$

Since $t=3$ satisfy all three eqs. we get $(x(3), y(3), z(3)) = (9,-8,28)$

iii) For $(4, 7, -6)$

$$\begin{cases} x = t^2 = 4 \Rightarrow t = 2 \text{ or } t = -2 \\ y = 1 - 3t = 7 \Rightarrow t = 2 \\ z = 1 + t^3 = -6 \end{cases}$$
Since $t = 2$ DOES NOT satisfy the final eq,

we can't solve this system, which means that the curve DOES NOT PASSES THROUGH THE POINT $(4, 7, -6)$.

---

47) Suppose the trajectories of two particles are given by the vector functions:

$$\vec{r}_1(t) = \langle t^2, 7t - 12, t^2 \rangle \quad ; \quad \vec{r}_2(t) = \langle 4t - 3, t^2, 5t - 6 \rangle$$

for $t \geq 0$. Do the particles collide?

We want to solve the following system of eqs:

$\vec{r}_1(t) = \vec{r}_2(t)$, for some t

$x_1(t) = t^2$; $x_2(t) = 4t - 3$

$y_1(t) = 7t - 12$; $y_2(t) = t^2$

$z_1(t) = t^2$; $z_2(t) = 5t - 6$

$$\begin{cases} t^2 = 4t - 3 \\ 7t - 12 = t^2 \\ t^2 = 5t - 6 \end{cases} \Leftrightarrow \begin{cases} t^2 - 4t + 3 = 0 \text{ (i)} \\ t^2 - 7t + 12 = 0 \text{ (ii)} \\ t^2 - 5t + 6 = 0 \text{ (iii)} \end{cases}$$

(i) $t^2 - 4t + 3 = 0 \Leftrightarrow t = \frac{4 \pm \sqrt{16 - 12}}{2 \cdot 1} \Leftrightarrow t = \frac{4 \pm \sqrt{4}}{2} \Leftrightarrow t = \frac{4 \pm 2}{2} \therefore \boxed{t = 3}$ or $\boxed{t = 1}$

(ii) $t^2 - 7t + 12 = 0 \Leftrightarrow t = \frac{7 \pm \sqrt{49 - 48}}{2 \cdot 1} \Leftrightarrow t = \frac{7 \pm 1}{2} \Leftrightarrow \boxed{t = 4}$ or $\boxed{t = 3}$

(iii) $t^2 - 5t + 6 = 0 \Leftrightarrow t = \frac{5 \pm \sqrt{25 - 24}}{2 \cdot 1} \Leftrightarrow t = \frac{5 \pm 1}{2} \Leftrightarrow \boxed{t = 3}$ or $\boxed{t = 2}$

Since $t = 3$ satisfy all three equations, the particles Do collide.

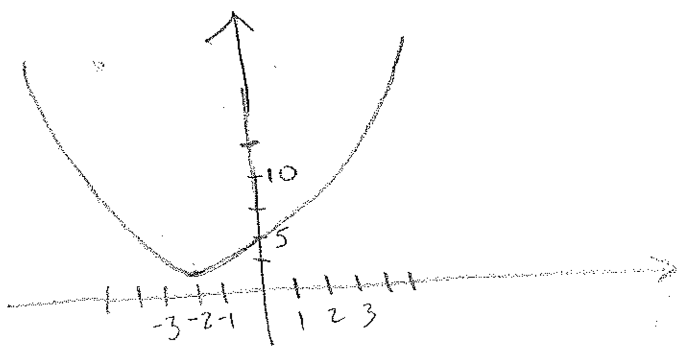
In particular, they collide at $t = 3$ at point $(9, 9, 9)$ since:

$$\vec{r}_1(3) = \langle 9, 9, 9 \rangle = \vec{r}_2(3)$$

SECTION 13.2 :

(3) (a) Sketch the plane curve with the given vector eq.

$\vec{r}(t) = \langle t-2, t^2+1 \rangle$



t	x(t)	y(t)
-3	-5	10
-2	-4	5
-1	-3	2
0	-2	1
1	-1	2
2	0	5
3	5	10

Eliminating the parameter t.

$x(t) = t-2$; $y(t) = t^2+1$

$\Downarrow$

$t = x+2 \Rightarrow y(x) = (x+2)^2 + 1 = x^2+4x+4+1$

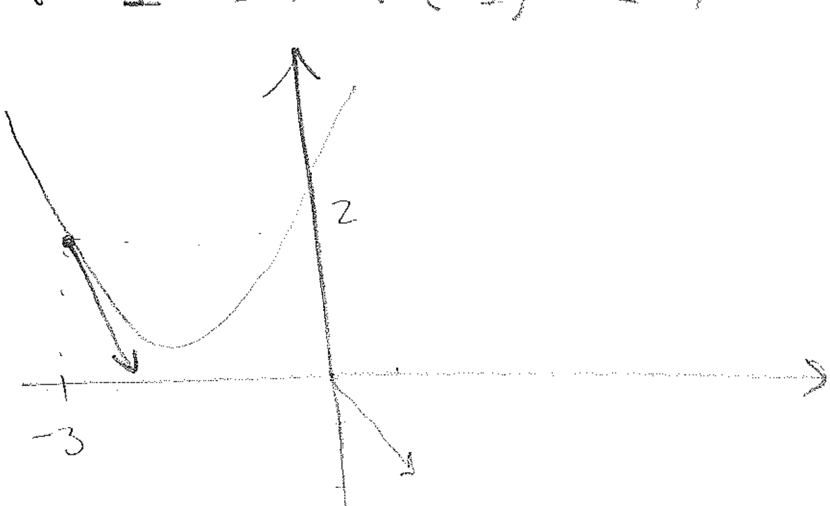
$\boxed{= x^2+4x+5}$

(b) Find $\vec{r}'(t)$

$\vec{r}'(t) = \langle (t-2)', (t^2+1)' \rangle = \langle 1, 2t \rangle$

(c) Sketch the position vector $\vec{r}(t)$ and the tangent vector $\vec{r}'(t)$ for the given value of t.

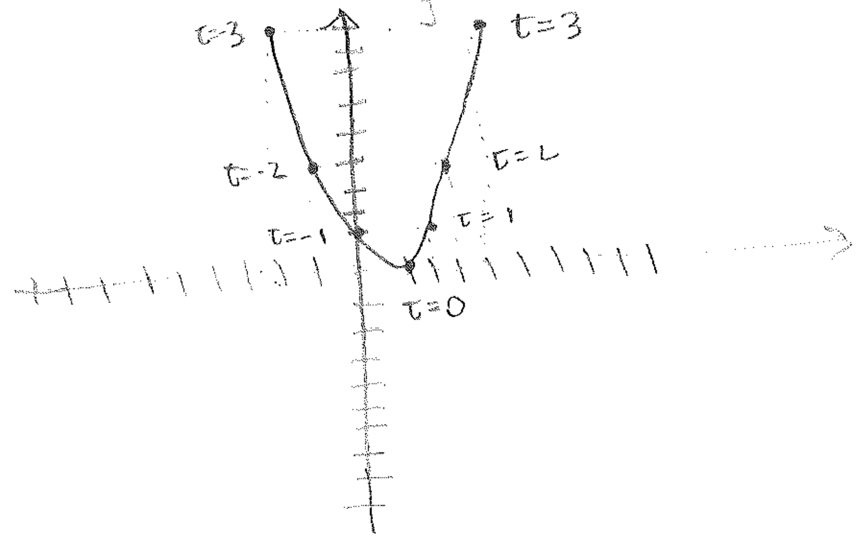
$t=-1 \Rightarrow \vec{r}'(-1) = \langle 1, -2 \rangle$



Similar to 13.2 (3–8).

$$\vec{r}(t) = (1+t)\hat{i} + t^2\hat{j}$$

(a) Sketch the plane curve.



t	$x(t)=1+t$	$y(t)=t^2$
-3	-2	9
-2	-1	4
-1	0	1
0	1	0
1	2	1
2	3	4
3	4	9

Eliminating the variable t:

$x(t) = 1+t$; $y(t) = t^2$

$\Rightarrow t = x-1$

$\Rightarrow y(x) = (x-1)^2$

$$\boxed{y(x) = x^2 - 2x + 1}$$

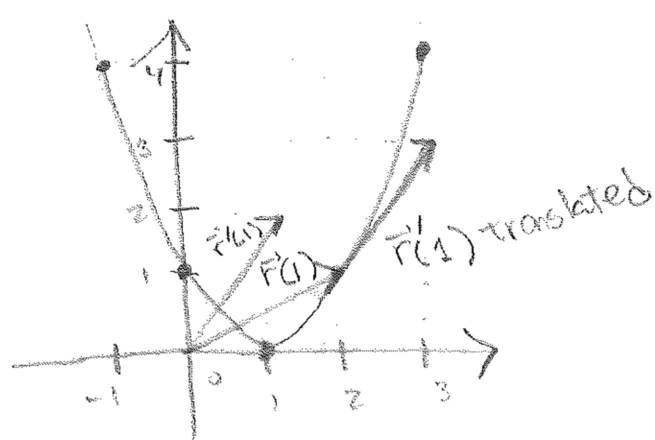
(b) Find $\vec{r}'(t)$

$\vec{r}'(t) = \langle (1+t)', (t^2)' \rangle = \langle 1, 2t \rangle$

(c) Sketch the position vector $\vec{r}(t)$ and the tangent vector $\vec{r}'(t)$ for $t = 1$.

If $t = 1$, then $\vec{r}'(1) = \langle 1, 2 \rangle$

$\vec{r}(1) = \langle 2, 1 \rangle$



16) Find the derivative of the vector function:

$\vec{r}(t) = t\vec{a} \times (\vec{b} + t\vec{c})$

$\vec{r}'(t) = \vec{a} \times (\vec{b} + t\vec{c}) + t\vec{a} \times \vec{c}$

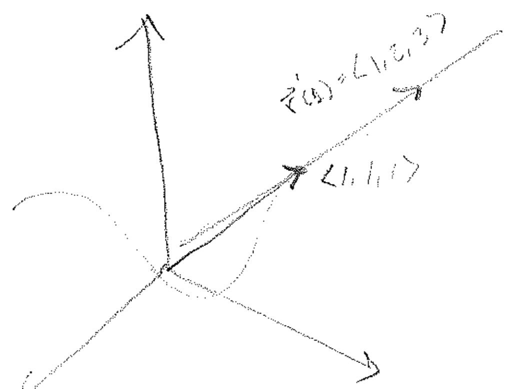
13) $\vec{r}(t) = e^{t^2}\hat{i} - \hat{j} + \ln(1+3t)\hat{k}$

$$\boxed{\vec{r}'(t) = 2te^{t^2}\hat{i} + \left(\frac{3}{1+3t}\right)\hat{k}}$$

Similar to (23-26)

Find parametric equations for the tangent line to the curve

$x(t) = t$; $y(t) = t^2$; $z(t) = t^3$ at $(1,1,1)$.



$\vec{r}(t) = \langle t, t^2, t^3\rangle$

$\Rightarrow \vec{r}'(t) = \langle 1, 2t, 3t^2\rangle$.

the direction of the line tangent to the curve at $(1,1,1)$ is given by the vector $\vec{r}'(t)$, where

$\vec{r}(t) = (1,1,1)$. Hence, $t = 1$. the direction is:

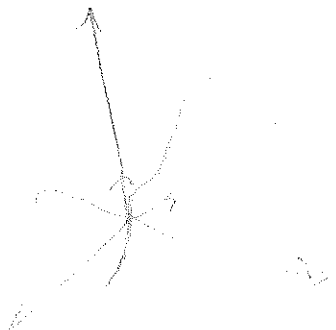
$\vec{r}'(1) = \langle 1,2,3\rangle = \vec{v}$.

the equation for the tangent line is:

$\vec{l}(s) = \vec{r_0} + s\vec{v}$

$$\Rightarrow \boxed{\begin{aligned}\vec{l}(s) &= \langle 1,1,1\rangle + s\langle 1,2,3\rangle \\ \vec{l}(s) &= (1+s)\hat{i} + (1+2s)\hat{j} + (1+3s)\hat{k}\end{aligned}}$$

33) the curves $\vec{r_1}(t) = \langle t, t^2, t^3 \rangle$ and $\vec{r_2}(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin. Find their angle of intersection.



$$\vec{r_1}'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$\vec{r_2}'(t) = \langle \cos t, 2 \cdot \cos(2t), 1 \rangle$$

AT $(0,0,0)$ $t = 0$.

Hence, the vectors tangent to the curves at the origin are

$$\vec{r_1}'(0) = \langle 1, 0, 0 \rangle$$

$$\vec{r_2}'(0) = \langle 1, 2, 1 \rangle.$$

the angle between these is

$$\vec{r_1}'(0) \cdot \vec{r_2}'(0) = \langle 1, 0, 0 \rangle \cdot \langle 1, 2, 1 \rangle = 1$$

$$\vec{r_1}'(0) \cdot \vec{r_2}'(0) = \sqrt{1}\,\sqrt{6}\,\cos\theta = \sqrt{6} \cdot \cos\theta$$

Hence, $\theta = \arccos\left(\frac{1}{\sqrt{6}}\right) \approx \boxed{66^\circ}$

35) Evaluate the integral

$$\int_0^2 (t\hat{i} - t^3\hat{j} + 3t^5\hat{k})\,dt = \int_0^2 t\hat{i}\,dt - \int_0^2 t^3\hat{j}\,dt + 3\int_0^2 t^5\hat{k}\,dt$$

$$= \left[\frac{t^2}{2}\right]_0^2 \hat{i} - \left[\frac{t^4}{4}\right]_0^2 \hat{j} + 3\left[\frac{t^6}{6}\right]_0^2 \hat{k}$$

$$= \left(\frac{2^2}{2} - \cancel{\frac{0}{2}}\right)\hat{i} - \left(\frac{2^4}{4} - \cancel{\frac{0}{4}}\right)\hat{j} + 3\left(\frac{2^6}{6} - \cancel{\frac{0}{6}}\right)\hat{k}$$

$$= \boxed{2\hat{i} - 4\hat{j} + 32\hat{k}}$$

Similar to (13.3)(1-6)

Section 13.3

(i) Find the length of the curve

$$\vec{r}(t) = \langle \sin t, \cos t, t \rangle, \quad 0 \leq t \leq 4\pi$$

the length is

$$\int_0^{4\pi} \sqrt{(\sin'(t))^2 + (\cos'(t))^2 + (t')^2}\, dt$$

$$= \int_0^{4\pi} \sqrt{\cos^2(t) + (-\sin(t))^2 + 1^2}\, dt$$

$$= \int_0^{4\pi} \sqrt{\underbrace{(\cos^2(t) + \sin^2(t))}_{1} + 1}\, dt = \int_0^{4\pi} \sqrt{1+1}\, dt$$

$$= \int_0^{4\pi} \sqrt{2}\, dt = \sqrt{2} \int_0^{4\pi} dt = \sqrt{2}\,[4\pi - 0] = \boxed{4\sqrt{2}\,\pi}$$

(ii) $\vec{r}(t) = \langle 2t, \ln t, t^2 \rangle, \quad 1 \leq t \leq e.$

the length is

$$\int_1^{e} \sqrt{(2t')^2 + (\ln t')^2 + (t^{2\prime})^2}\, dt$$

$$= \int_1^{e} \sqrt{2^2 + \left(\frac{1}{t}\right)^2 + (2t)^2}\, dt = \int_1^{e} \sqrt{4 + \frac{1}{t^2} + 4t^2}\, dt$$

$$= \int_1^{e} \sqrt{\frac{4t^2+1+4t^4}{t^2}}\, dt = \int_1^{e} \sqrt{\frac{(2t^2+1)^2}{t^2}}\, dt = \int_1^{e} \frac{2t^2+1}{t}\, dt$$

$$= \int_1^{e} 2t\, dt + \int_1^{e} \frac{1}{t}\, dt = \left[t^2\right]_1^{e} + \left[\ln(t)\right]_1^{e} = e^2 - 1 + \ln(e) - \ln(1)$$

$$= \boxed{e^2}$$

Section 13.4

(9) Find the velocity, acceleration, and speed of a particle with the given position function

$\vec{r}(t) = \langle t^2+t,\ t^2-t,\ t^3 \rangle$

Velocity$(t) = \vec{r}\,'(t)$; speed $= |\vec{r}\,'(t)|$

acceleration$(t) = \vec{r}\,''(t)$.

Velocity: $\vec{v}(t) = \vec{r}\,'(t) = \langle (t^2+t)', (t^2-t)', (t^3)' \rangle$

$$\boxed{\vec{v}(t) = \langle 2t+1,\ 2t-1,\ 3t^2 \rangle}$$

Speed: $|\vec{v}(t)| = \sqrt{(2t+1)^2 + (2t-1)^2 + (3t^2)^2}$

$= \sqrt{4t^2+4t+1+4t^2-4t+1+9t^4}$

$$\boxed{\text{speed}(t) = \sqrt{9t^4+8t^2+2}}$$

acceleration:

$\vec{v}\,'(t) = \langle (2t+1)', (2t-1)', (3t^2)' \rangle$

$$\boxed{\text{acceleration}(t) = \langle 2, 2, 6t \rangle}$$

15 Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

$\vec{a}(t) = \hat{\imath} + 2\hat{\jmath}$, $\vec{v}(0) = \hat{k}$; $\vec{r}(0) = \hat{\imath}$

By definition, <u>the velocity</u> is the integral of the acceleration:

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \hat{\imath} + 2\hat{\jmath}\,dt$$
$$= t\hat{\imath} + 2t\hat{\jmath} + \vec{C}, \text{ where } \vec{C} \text{ is a constant vector.}$$

To find the value of $\vec{C}$, plug in the initial condition for $\vec{v}(t)$:

$$\vec{v}(0) = \hat{k} = 0\hat{\imath} + 2(0)\hat{\jmath} + \vec{C} \Rightarrow \vec{C} = \hat{k}$$,

from which it follows that the velocity function is:

$$\boxed{\vec{v}(t) = t\hat{\imath} + 2t\hat{\jmath} + \hat{k}}$$

By definition, the <u>position</u> is the integral of the velocity:

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int t\hat{\imath} + 2t\hat{\jmath} + \hat{k}\,dt$$
$$= \frac{t^2}{2}\hat{\imath} + t^2\hat{\jmath} + t\hat{k} + \vec{C}, \text{ where } \vec{C} \text{ is constant}$$

To find the value of $\vec{C}$, plug in the initial condition for $\vec{r}(t)$:

$$\vec{r}(0) = \hat{\imath} = \frac{0^2}{2}\hat{\imath} + 0^2\hat{\jmath} + 0\hat{k} + \vec{C} \Rightarrow \hat{\imath} = \vec{C}$$,

the function is $\boxed{\vec{r}(t) = \frac{t^2+1}{2}\hat{\imath} + t^2\hat{\jmath} + t\hat{k}}$

Similar to (13.4) (19)

The position fnction of a particle is given by

$$\vec{r}(t) = \langle t^2, 5t, t^2+2t \rangle$$

At what time is the speed minimum?

By definition, the speed is the magnitude of the velocity. The velocity $\vec{v}(t)$ is the derivative of the position:

$$\vec{v}(t) = \vec{r}\,'(t) = \langle (t^2)', (5t)', (t^2+2t)' \rangle$$

$$\Rightarrow \quad \vec{v}(t) = \langle 2t, 5, 2t+2 \rangle$$

the speed is therefore $|\vec{v}(t)|$.

$$|\vec{v}(t)| = \sqrt{(2t)^2 + (5)^2 + (2t+2)^2}$$

$$= \sqrt{4t^2 + 25 + 4t^2 + 8t + 4}$$

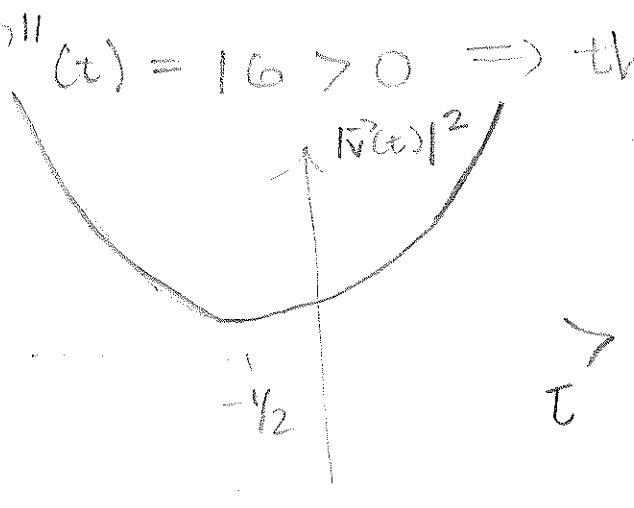
$$|\vec{v}(t)| = \sqrt{8t^2 + 8t + 29}$$

To minimize this function is equivalent to minimize the square $f(t) = |\vec{v}(t)|^2 = 8t^2 + 8t + 29$. this is a function on one variable t. Minimize it:

since $f(t)$ acquires a minimum in the same place as $f'(t)$

$$f'(t) = 16t + 8 = 0 \Rightarrow t = -\frac{1}{2}$$

$$f''(t) = 16 > 0 \Rightarrow \text{the point is an absolute minimum}$$



At $t = -1/2$, the speed is

$$|\vec{v}(-1/2)| = \sqrt{\frac{8}{4} - \frac{8}{2} + 29}$$

$$= \sqrt{2 - 4 + 29} = \boxed{\sqrt{27}}$$

Section 14.1 :

(3) $P(L,K) = 1.47 L^{0.65} K^{0.35}$, $L = \#$ of labor hours (in thousands)
K = invested capital (in millions of \$)
$P(L,K)$ = monetary value of the entire production in millions of \$

Find $P(120,20)$ and interpre it.

Solution: $P(L=120, K=20) = 1.47(120)^{0.65}(20)^{0.35} = \boxed{94.2205518}$

If we input 120,000 hours of labor and 20,000,000 \$, we obtain a value of 94 million \$ for the entire production.

(5) $S = f(w,h) = 0.1091\, w^{0.425} h^{0.725}$, w = weight (in pounds)
h = height (in inches)
S is measured in square feet.

(a) Find $f(160,70)$ and interpret it

$f(160,70) = 0.1091\,(160)^{0.425}(70)^{0.725} = \boxed{20.5244639 \text{ sqft.}}$

A Body with weight 160 pound and height 70 inches, will have a surface area of 20.5 sqft.

(b) —

(7) $h = f(v,t)$ has value in table 4.

(a) $f(40,15) = 25$, a speed of 40 for 15 hours produces wave height of 25.

(b) $h = f(30,t)$, this describe the height of waves for which the speed of the wind is 30 for a given time t.

It increases from 9 to 19 as t increases from 5 to 50

(9) Let $g(x,y) = \cos(x+2y)$

(a) Evaluate $g(2,-1) = \cos(2+2(-1)) = \cos(2-2) = \cos(0) = \boxed{1}$

(b) Find the domain of g.

$\cos(w)$ is defined for all values of $w \in \mathbb{R}$.

Hence, $w = x+2y$, the domain is $(x,y) \in \mathbb{R}^2 = \mathbb{R}^2$.

(c) Find the range of g.

the range of $g(x,y)$ is the range of cos, which is $[-1,1]$.

(11) Let $f(x,y,z) = \sqrt{x} + \sqrt{y} + \sqrt{z} + \ln(4-x^2-y^2-z^2)$.

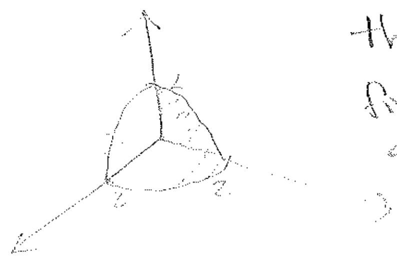
(a) Evaluate $f(1,1,1) = \sqrt{1} + \sqrt{1} + \sqrt{1} + \ln(4-1^2-1^2-1^2) = 3 + \ln(1) = \boxed{3}$

(b) Find and describe the domain of f.

$f(x,y,z)$ is defined iff

$x \geq 0$ and $y \geq 0$ and $z \geq 0$ and $4-x^2-y^2-z^2 > 0$

$\Rightarrow 4 > x^2+y^2+z^2$

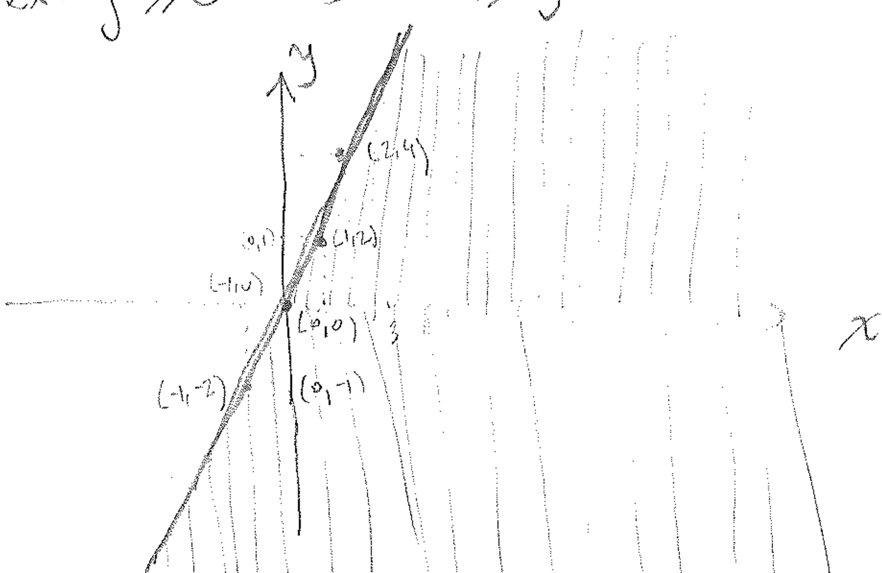


the interior of the first octant of the sphere of radius 2.

13) Find and sketch the domain of the function

$f(x,y) = \sqrt{2x-y}$, is defined when,

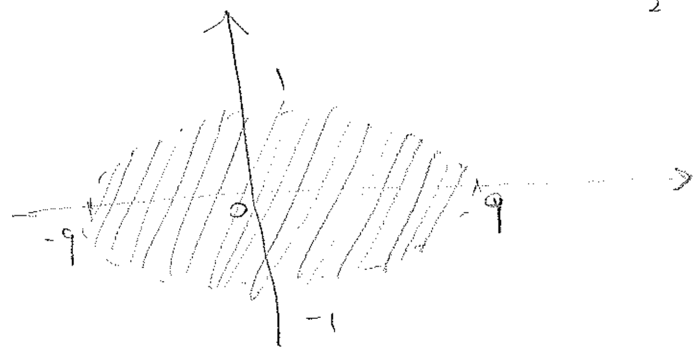
$2x - y \geq 0 \Leftrightarrow 2x \geq y \Leftrightarrow$ $2x = y$ or $2x > y$



(15) $f(x,y) = \ln(9 - x^2 - 9y^2)$ is defined if

$$9 - x^2 - 9y^2 > 0 \iff 9 > x^2 + 9y^2$$

$$1 > \frac{1}{9}x^2 + y^2$$



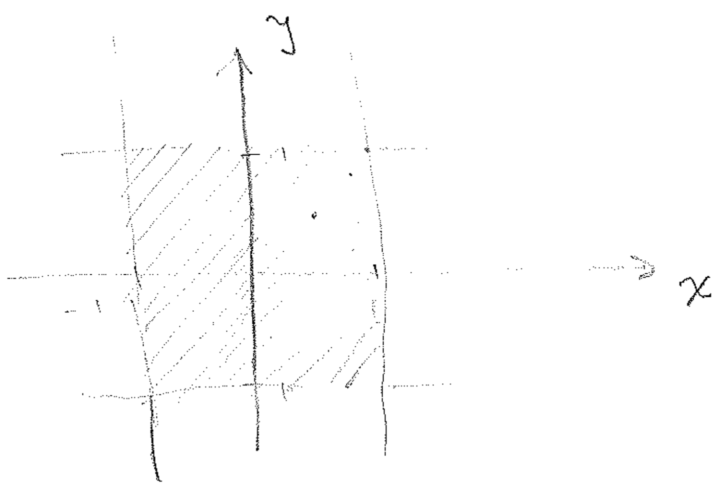
(17) $f(x,y) = \sqrt{1-x^2} - \sqrt{1-y^2}$ is define if

$$1 - x^2 \geq 0 \quad \text{and} \quad 1 - y^2 \geq 0$$

$$1 \geq x^2 \quad \text{and} \quad 1 \geq y^2$$

$(1 = x^2$ or $1 > x^2)$ and $(1 = y^2$ or $1 > y^2)$

$x = 1$ $x = -1$ $y = 1$ $y = -1$



(14) $f(x,y) = \sqrt{xy}$, f is defined iff $xy \geq 0$, i.e.
$x \cdot y = 0$ OR $x \cdot y > 0$.
From the first equation:

$$x \cdot y = 0 \implies x = 0 \text{ OR } y = 0$$

From the second equation:

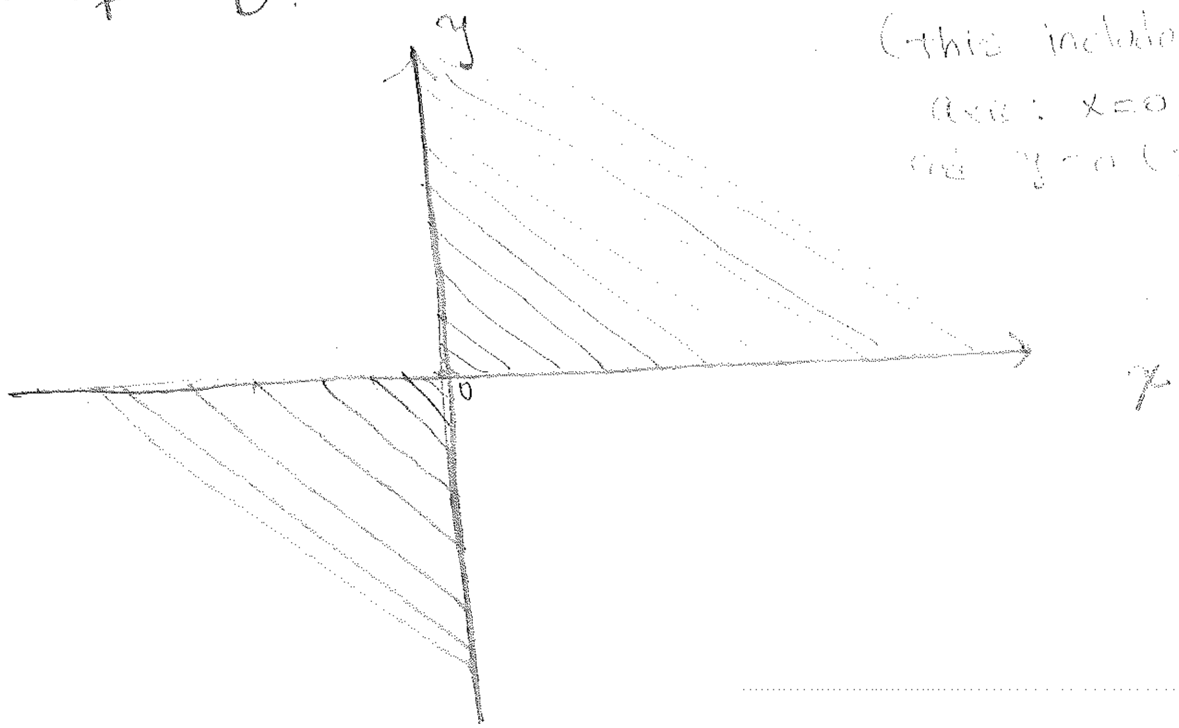
$$x \cdot y > 0 \implies (x > 0 \text{ AND } y > 0) \text{ OR } (x < 0 \text{ AND } y < 0).$$

the Domain is

$$D = \{(x,y) : (x = 0 \text{ OR } y = 0) \text{ OR } (x > 0 \text{ AND } y > 0) \text{ OR } (x < 0 \text{ AND } y < 0)\}$$

Graph of D:

(this include the axis: $x=0$ (y-axis) and $y=0$ (x-axis)).



(28) Sketch the graph of the function:

$f(x,y) = 1 + 2x^2 + 2y^2$

TRACES:

(i) In the xy-plane

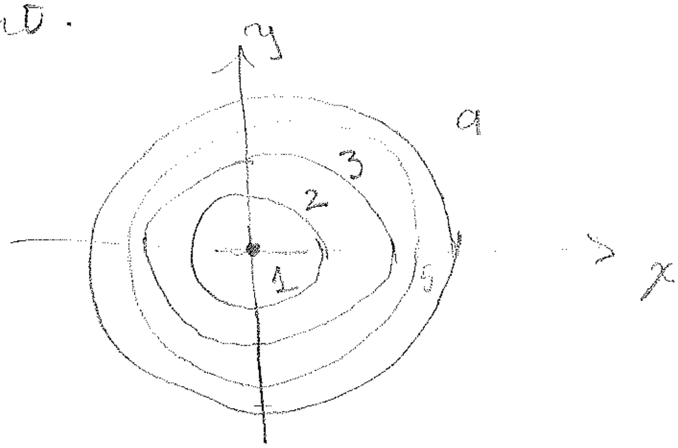
$f(x,y) = 1 + 2x^2 + 2y^2 = K$, for K constant.

$\Rightarrow 2x^2 + 2y^2 = K - 1$

$x^2 + y^2 = \dfrac{K-1}{2}$,

this is a circle of radius $r^2 = \dfrac{K-1}{2} \Leftrightarrow r = \sqrt{\dfrac{K-1}{2}}$,

Hence, $\dfrac{K-1}{2} \geq 0$, $K - 1 \geq 0$, $K \geq 1$. If $K = 1$, then this is a point.



(ii) In the yz-plane

$f(K,y) = 1 + 2K^2 + 2y^2$, $g(y) = (1 + 2K^2) + 2y^2$

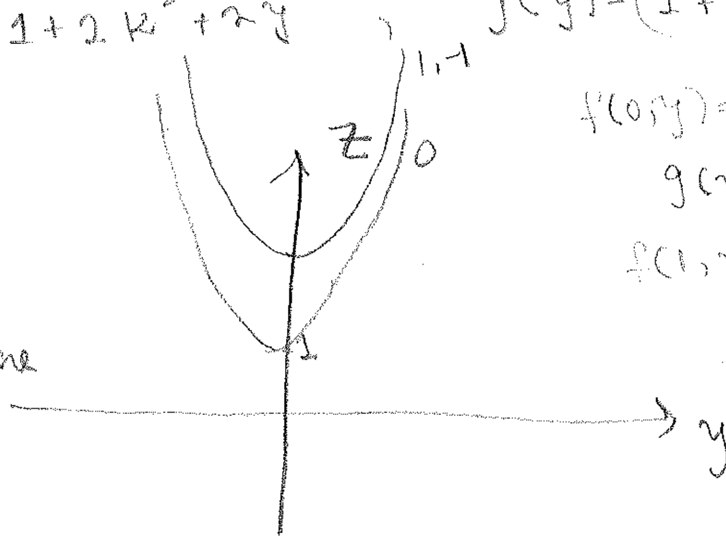
$f(0,y) = 1 + 2y^2$

$g(y) = 2y^2 + 1$

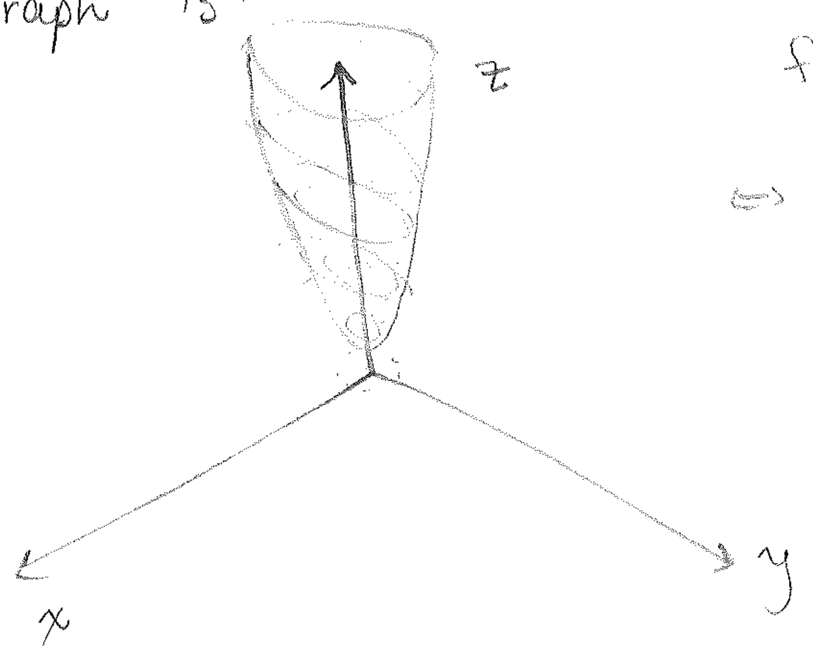
$f(1,y) = 3 + 2y^2$

$f(-1,y) = 3 + 2y^2$

the xz-plane is symmetric to the yz-plane



the graph is:



$$f(x,y) = 1 + 2x^2 + 2y^2$$

$$\Leftrightarrow \quad z = 2x^2 + 2y^2 + 1$$

$$\frac{z}{2} = x^2 + y^2 + \frac{1}{2}$$

$$\frac{z}{2} - \frac{1}{2} = x^2 + y^2$$

$$\frac{z-1}{2} = x^2 + y^2$$

---

(44) Draw a contour map of the function showing several level curves.

$$f(x,y) = x^3 - y.$$

By definition, the level curves are:

$$f(x,y) = x^3 - y = k, \quad \text{for } k \text{ a constant.}$$

$k=0 \Rightarrow x^3 - y = 0 \Rightarrow y = x^3.$

$k=1 \Rightarrow x^3 - y = 1 \Rightarrow y = x^3 - 1$

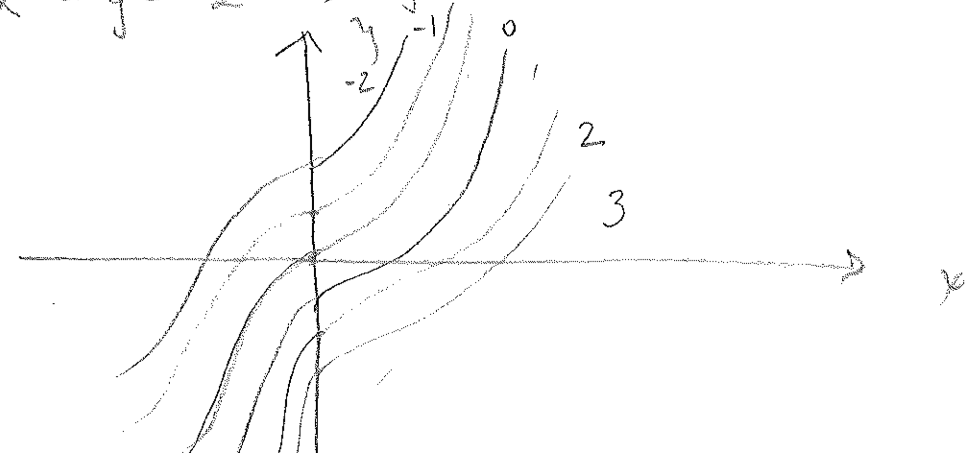
$k=2 \Rightarrow x^3 - y = 2 \Rightarrow y = x^3 - 2$

$\vdots$

$k=-1 \Rightarrow x^3 - y = -1 \Rightarrow y = x^3 + 1$

$k=-2 \Rightarrow x^3 - y = -2 \Rightarrow y = x^3 + 2$

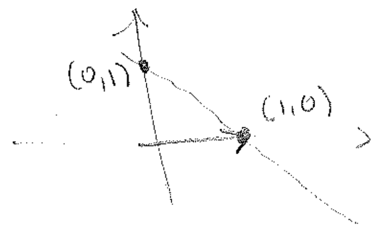
$\vdots$



Section 14.2:

(8) Find the limit, if it exists, or show that the limit does not exists.

$$\lim_{(x,y)\mapsto(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right).$$



$m=-1$

Let us approach the limit when $\boxed{y=1-x}$

$y=mx+b$

$$\begin{cases} 0=m+b=m+1 \\ 1=0+b \Rightarrow b=1 \end{cases}$$

$\Rightarrow y=-x+1$

$\Rightarrow \boxed{y=1-x}$

$$\lim_{x\to 1} \ln\left(\frac{1+(1-x)^2}{x^2+x(1-x)}\right)$$

$$=\lim_{x\to 1} \ln\left(\frac{1+1-2x+x^2}{x^2+x-x^2}\right)$$

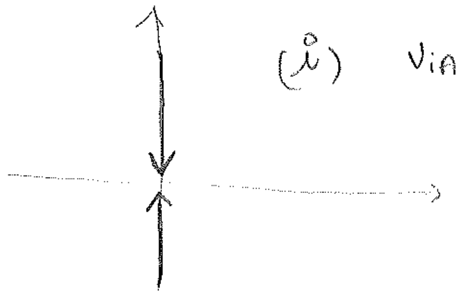
$$=\lim_{x\to 1} \ln\left(\frac{x^2-2x+2}{x}\right) = \ln\left(\frac{1-2+2}{1}\right) = \ln(1) = 0$$

$$\boxed{\lim_{(x,y)\mapsto(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right) = \ln\left(\frac{1+0}{1+0}\right) = \ln(1) = 0}$$

(10) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2(x)}{x^4+y^4}$

<u>Solution</u>:

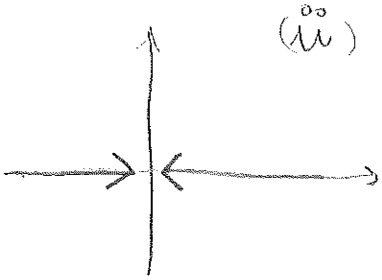
Let us approach $(0,0)$ in two different paths.



(i) via the y-axis.

then $x=0$: $\quad f(0,y) = \dfrac{5y^4\cdot\cos^2(0)}{0^4+y^4} = \dfrac{5y^4}{y^4} = 5$

$\Rightarrow \displaystyle\lim_{y\to 0} f(0,y) = 5$



(ii) via the x-axis

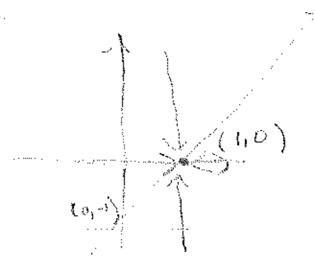
then $y=0$: $\quad f(x,0) = \dfrac{5(0)^4\cos^2(x)}{x^4+(0)^4} = \dfrac{0}{x^4} = 0$

$\Rightarrow \displaystyle\lim_{x\to 0} f(x,0) = 0$

Since f has two different limits along two different paths, the limit does not exists.

---

(12) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2} = \frac{xy-y}{x^2-2x+1+y^2}$



<u>Solution</u>: Let us approach $(1,0)$ in two different paths:

(i) $y=0 \Rightarrow$ limit $= 0$

(ii) the line through: $(0,-1)$; $(1,0)$. $\quad y = mx+b$

$\begin{cases} 0 = m+b \\ -1 = 0+b \Rightarrow b=-1 \end{cases}$ $\quad 0 = m-1 \Rightarrow m=1 \quad \Rightarrow \boxed{y = x-1}$

$f(x,x-1) = \dfrac{x(x-1)-(x-1)}{(x-1)^2+(x-1)^2}$

$= \dfrac{x^2-x-x+1}{2(x-1)^2}$

$= \dfrac{x^2-2x+1}{2(x-1)^2} \quad \Rightarrow$ limit $= \dfrac{1}{2}$

$= \dfrac{(x-1)^2}{2(x-1)^2} = \dfrac{1}{2}$

Since f has two different limits along two different paths, the limit does not exists.

14) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-y^4}{x^2+y^2}$

$(x-y)^2(x+y)^2$

$(x^2-y^2)^2$

Approach via: $y=mx$

$$f(x,mx) = \frac{x^4-(mx)^4}{x^2+(mx)^2} = \frac{x^4(1-m^4)}{\cancel{x^2}(1+m^2)} = \frac{x^2(1-m^4)}{(1+m^2)}$$

All paths approach zero ....

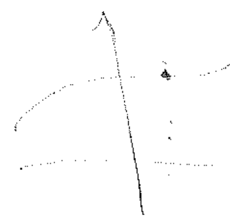
$(x^2+y^2)(x^2-y^2)$

Note that

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} \quad \lim_{(x,y)\to(0,0)} \frac{\cancel{(x^2+y^2)}(x^2-y^2)}{\cancel{(x^2+y^2)}} \quad \lim_{(x,y)\to(0,0)} x^2-y^2 = \boxed{0}$$

(38) Determine the set of points at which the function is continuous.

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+xy+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$



A function is continuous at (x,y) if $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

this function is a ratio of polynomial and hence, the limit exists everywhere where it is defined. the only issue could be at $(0,0)$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2}$$

Two paths:

(i) $x=0$, $\dfrac{0}{y^2}=0$

(ii) $x=y$ $\dfrac{x^2}{x^2+x^2+x^2} = \dfrac{\cancel{x^2}}{3\cancel{x^2}} = 1/3$

$\Rightarrow$ the limit DNE. the function is not continuous at $(0,0)$. It is continuous at $\mathbb{R}^2 \setminus \{(0,0)\}$

(37) $f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$

the only potential issue is at $(0,0)$.

$$\lim_{(x,y)\mapsto(0,0)} f(x,y) = \lim_{(x,y)\mapsto(0,0)} \frac{x^2y^3}{2x^2+y^2}$$

<u>Two paths</u>:

(i) $x=0 \quad = \lim_{y\to 0} \dfrac{0}{y^2} = \boxed{0}$

(ii) $x=y \implies \dfrac{x^5}{2x^2+x^2} = \dfrac{x^5}{3x^2} = \dfrac{x^3}{3} = \boxed{0}$

these paths show that $\lim_{(x,y)\mapsto(0,0)} f(x,y) \neq 1$. Hence, the function is not continuous at $(0,0)$.

$\boxed{\text{the function is continuous at } \mathbb{R}^2 \setminus \{(0,0)\}.}$

Section 14.3:

(3) (a) $f_T(-15,30) \rightarrow$ the change in Temperature at $(-15,30)$

$$g(V) = f_T(-15, V)$$

$$g'(-15) = \lim_{h\to 0} \frac{g(-15+h)-g(-15)}{h} = \lim_{h\to 0} \frac{f(-15+h,30)-f(-15,30)}{h}$$

Approximate using $h=-5$ and $h=5$

$$h=-5: \ g(-20) \approx \frac{g(-20)-g(-15)}{-5} = \frac{f(-20,30)-f(-15,30)}{-5} = \frac{-33+26}{-5} = \frac{+7}{5}$$

$$h=5: \ g(-10) \approx \frac{g(-10)-g(-15)}{5} = \frac{f(-10,30)-f(-15,30)}{5} = \frac{-20+26}{5} = \frac{6}{5}$$

Average values: $\dfrac{\frac{7}{5}+\frac{6}{5}}{2} = \dfrac{\frac{13}{5}}{\frac{2}{1}} = \dfrac{13}{10} = \boxed{1,3}$

(b) Sign of $\frac{\partial W}{\partial T}$ positive ; Sign of $\frac{\partial W}{\partial V}$ negative

(c) 0

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(5) (a) $f_x(1,2)$ positive (b) $f_y(1,2)$ negative

---

(15) Find the first partial derivatives of the function

$f(x,y) = y^5 - 3xy$.

$f_x(x,y) = -3y$; $f_y(x,y) = 5y^4 - 3x$

(17) $f(x,t) = e^{-t}\cos(\pi x)$

$f_x(x,t) = -e^{-t}\pi\sin(\pi x)$; $f_t(x,t) = -e^{-t}\cos(\pi x)$.

(19) $z = (2x+3y)^{10} = f(x,y)$

$f_x(x,y) = 20(2x+3y)^9$; $f_y(x,y) = 30(2x+3y)^9$

(25) $g(u,v) = (u^2v - v^3)^5$

$g_u(u,v) = (5(u^2v-v^3)^4)(2uv) = 10uv(u^2v-v^3)^4$

$g_v(u,v) = (5(u^2v-v^3)^4)(u^2-3v^2) = 5(u^2-3v^2)(u^2v-v^3)^4$

(29) $F(x,y) = \int_y^x \cos(e^t)\,dt \Rightarrow \quad (F(x,y))' = \left[\cos(e^t)\right]_y^x = \cos(e^x) - \cos(e^y)$

$F_x(x,y) = \cos(e^x) \quad ; \quad F_y(x,y) = -\cos(e^y)$

(31) $f(x,y,z) = xz - 5x^2y^3z^4$

$f_x(x,y,z) = z - 10xy^3z^4, \quad f_y(x,y,z) = -15x^2y^2z^4$

$f_z(x,y,z) = x - 20x^2y^3z^3$

(41) Find the indicated partial derivative.

$f(x,y) = \ln(x+\sqrt{x^2+y^2}); \quad f_x(3,4)$.

$$f_x(x,y) = \left(\frac{1}{x+\sqrt{x^2+y^2}}\right)\cdot\left(1+\frac{1}{\cancel{2}}\frac{\cancel{2}x}{\sqrt{x^2+y^2}}\right)$$

$$f_x(3,4) = \left(\frac{1}{3+\sqrt{9+16}}\right)\cdot\left(1+\frac{3}{\sqrt{9+16}}\right) = \frac{1}{8}\left(1+\frac{3}{5}\right)$$

$$= \frac{1}{\cancel{8}}\left(\frac{\cancel{8}}{5}\right) = \boxed{\frac{1}{5}}$$

(45) $f(x,y) = xy^2 - x^3y$ $\qquad f_x(x,y) = y^2 - 3x^2y$

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0}\frac{\left[(x+h)y^2-(x+h)^3y\right]-xy^2+x^3y}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{xy^2}+hy^2-\cancel{x^3y}-3x^2hy-3xh^2y-h^3-\cancel{xy^2}+\cancel{x^3y}}{h}$$

$$= \lim_{h\to 0}\frac{hy^2-3x^2hy-h^3}{h} = \lim_{h\to 0}\frac{\cancel{h}(y^2-3x^2y-h^2)}{\cancel{h}} = \lim_{h\to 0} y^2-3x^2y-h^2 = \boxed{y^2-3x^2y}$$

55 Find all the second partial derivatives

$w = \sqrt{u^2+v^2}$ $\quad \frac{\partial w}{\partial u}, \frac{\partial^2 w}{\partial u^2}, \frac{\partial w}{\partial v}; \frac{\partial^2 w}{\partial v^2}, \frac{\partial w}{\partial u \partial v} = \frac{\partial w}{\partial v \partial u}$

$$\frac{\partial^2 w}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{1(\cancel{2}u)}{\cancel{2}\sqrt{u^2+v^2}}\right) = \frac{\partial}{\partial u}\left(\frac{u}{\sqrt{u^2+v^2}}\right)$$

$$= \frac{1}{(u^2+v^2)^{1/2}} + \left(\frac{-1}{\cancel{2}}\right)\frac{u \cdot \cancel{2}u}{(u^2+v^2)^{3/2}} = \frac{1}{(u^2+v^2)^{1/2}} - \frac{u^2}{(u^2+v^2)^{3/2}}$$

$$= \frac{u^2+v^2-u^2}{(u^2+v^2)^{3/2}} = \boxed{\frac{v^2}{(u^2+v^2)^{3/2}}}$$

$$\frac{\partial^2 w}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial v}\right) = \frac{\partial}{\partial v}\left(\frac{1(\cancel{2}v)}{\cancel{2}\sqrt{u^2+v^2}}\right) = \frac{\partial}{\partial v}\left(\frac{v}{\sqrt{u^2+v^2}}\right)$$

$$= \frac{1}{\sqrt{u^2+v^2}} + \left(-\frac{1}{\cancel{2}}\right)\frac{v(\cancel{2}v)}{(u^2+v^2)^{3/2}} = \frac{1}{(u^2+v^2)^{1/2}} - \frac{v^2}{(u^2+v^2)^{3/2}}$$

$$= \frac{u^2+\cancel{v^2}-\cancel{v^2}}{(u^2+v^2)^{3/2}} = \boxed{\frac{u^2}{(u^2+v^2)^{3/2}}}$$

$$\frac{\partial^2 w}{\partial v \partial u} = \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right) = \frac{\partial}{\partial v}\left(\frac{u}{\sqrt{u^2+v^2}}\right) = \left(\frac{-1}{\cancel{2}}\right)\frac{u(\cancel{2}v)}{(u^2+v^2)^{3/2}}$$

$$= -\frac{uv}{(u^2+v^2)^{3/2}}$$

SECTION 14.4

(1) Find an equation of the tangent plane to the given surface at the specified point.

$z = 3y^2 - 2x^2 + x \quad , \quad (2,-1,-3)$

$\nabla f = \langle -4x+1, 6y, 1 \rangle$. the plane is given by

$$\nabla f(2,-1,-3) \cdot \langle x-2, y+1, z+3 \rangle = 0$$
$$\langle -7, -6, -1 \rangle \cdot \langle x-2, y+1, z+3 \rangle = 0$$
$$-7x + 14 - 6y - 6 - z - 3 = 0$$
$$-7x - 6y - z + 5 = 0$$
$$\boxed{7x + 6y + z = 5}$$

Similar to 1-6, find tangent plane:

$z = x^2 + 4y^2 \quad , \quad (2,1,8)$

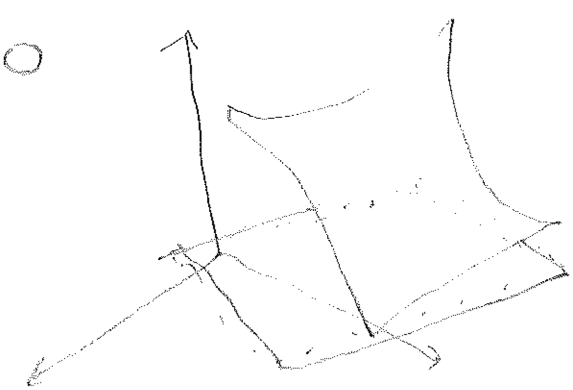
Up one dimension: $f(x,y,z) = x^2 + 4y^2 - z$

$\nabla f = \langle 2x, 8y, -1 \rangle$.

The gradient is perpendicular to the level surface!

the plane is:

$$\nabla f(2,1,8) \cdot \langle x-2, y-1, z-8 \rangle = 0$$
$$\langle 4, 8, -1 \rangle \cdot \langle x-2, y-1, z-8 \rangle = 0$$
$$4x - 8 + 8y - 8 - z + 8 = 0$$
$$\boxed{4x + 8y - z = 8}$$



Find the linear approximation to

$$z = 3 + \frac{x^2}{16} + \frac{y^2}{9} \quad \text{at } (-4,3)$$

Since $x=-4$, $y=3$ $\Rightarrow z = 5$, the point is $(-4,3,5)$

Let $f(x,y,z) = 3 + \frac{x^2}{16} + \frac{y^2}{9} - z$

then $\nabla f = \langle \frac{x}{8}, \frac{2}{9}y, -1 \rangle$

the plane is

$$\nabla f(-4,3,5) \cdot \langle x+4, y-3, z-5 \rangle = 0$$

$$\langle -\frac{1}{2}, \frac{2}{3}, -1 \rangle \cdot \langle x+4, y-3, z-5 \rangle = 0$$

$$-\frac{x}{2} \cancel{-2} + \frac{2}{3}y - 2 - z + 5 = 0$$

$$-\frac{x}{2} + \frac{2}{3}y - z + 1 = 0$$

$$\boxed{-\frac{1}{2}x + \frac{2}{3}y - z = -1}$$

(19) Given that f is differentiable function with $f(2,5)=6$, $f_x(2,5)=1$, and $f_y(2,5)=-1$, use linear approximation to estimate $f(2.2, 4.9)$.

Use tangent plane. Let $f(x,y,z)$ be a function of x, y, z.

then, $\nabla f \cdot \langle f_x, f_y, -1\rangle$

$$\nabla f(2,5,6) \cdot \langle x-2, y-5, z-6\rangle = 0$$

$$\langle f_x(2,5), f_y(2,5), -1\rangle \cdot \langle x-2, y-5, z-6\rangle = 0$$

$$\langle 1, -1, -1\rangle \cdot \langle x-2, y-5, z-6\rangle = 0$$

$$x-2+5-y-z+6=0$$

$$x-y-z+9=0 \Rightarrow \boxed{x-y-z=-9}$$

Now compute: $f(2.2, 4.9) \approx 2.2-4.9-z=-9$

$$\Rightarrow z = 2.2-4.9+9 = 4.1+2.2 = \boxed{6.3} \approx f(2.2, 4.9)$$

(17) Verify the linear approximation at $(0,0)$

$$\frac{2x+3}{4y+1} \approx 3+2x-12y. \qquad \frac{2(0)+3}{4(0)+1} = 3 = 3+2(0)-12(0).$$

Let $f(x,y,z) = \dfrac{2x+3}{4y+1} - z$ then $\nabla f(x,y,z) = \left\langle \dfrac{2}{4y+1}, \dfrac{2x+3}{4}, -1\right\rangle$

At $(0,0)$, we have $f(0,0)=3$. $\Rightarrow (0,0,3)$.

the tangent plane is:

$$\nabla f(0,0,3) \cdot \langle x, y, z-3\rangle = 0$$

$$\langle 2, \tfrac{3}{4}, -1\rangle \cdot \langle x, y, z-3\rangle = 0$$

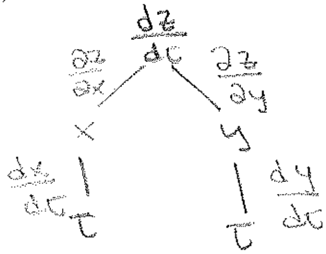
$$2x + \frac{3}{4}y - z + 3 = 0$$

$$8x + 3y - 4z + 3 = 0$$

Section 14.5:

(3) Use the Chain Rule to find dz/dt.

$z = \sqrt{1+x^2+y^2}$, $x(t) = \ln(t)$; $y(t) = \cos(t)$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}, \text{ where}$$

$$\frac{\partial z}{\partial x} = \left(\frac{1}{\cancel{2}}\right)\cdot\frac{1}{(1+x^2+y^2)^{1/2}}\cdot\cancel{2}x = \frac{x}{\sqrt{1+x^2+y^2}}$$

$$\frac{dx}{dt} = \frac{1}{t}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{1+x^2+y^2}}$$

$$\frac{dy}{dt} = -\sin(t)$$

Hence,

$$\frac{dz}{dt} = \frac{x}{t\sqrt{1+x^2+y^2}} - \frac{\sin(t)\, y}{\sqrt{1+x^2+y^2}}$$

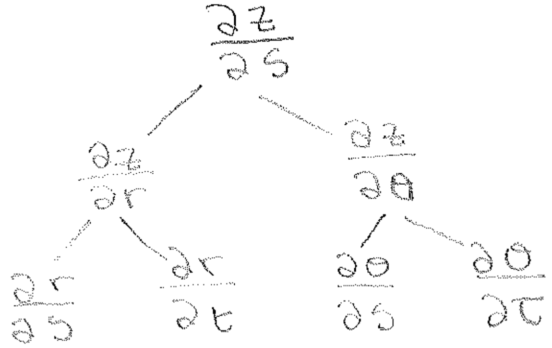
Now, replace definitions of x and y:

$$\frac{dz}{dt} = \frac{\ln(t)}{t\sqrt{1+\ln^2(t)+\cos^2(t)}} + \frac{\sin(t)\cos(t)}{\sqrt{1+\ln^2(t)+\cos^2(t)}}$$

$$\boxed{\frac{dz}{dt} = \frac{\ln(t) + t\sin(t)\cos(t)}{t\sqrt{1+\ln^2(t)+\cos^2(t)}}}$$

11. Use the Chain Rule to find to find $\partial z/\partial s$ and $\partial z/\partial t$.

$z = e^r \cdot \cos\theta$, $r = st$, $\theta = \sqrt{s^2+t^2}$.



Hence,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\,\frac{\partial \theta}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial t}$$

Compute each piece:

$$\frac{\partial z}{\partial r} = e^r \cdot \cos\theta \quad ; \quad \frac{\partial z}{\partial \theta} = -e^r \sin(\theta)$$

$$\frac{\partial r}{\partial s} = t \quad ; \quad \frac{\partial \theta}{\partial s} = \frac{s}{\sqrt{s^2+t^2}}$$

$$\frac{\partial r}{\partial t} = s \quad ; \quad \frac{\partial \theta}{\partial t} = \frac{t}{\sqrt{s^2+t^2}}$$

Combine results:

$$\frac{\partial z}{\partial s} = e^r \cdot \cos(\theta)\cdot t + (-e^r \cdot \sin(\theta))\,\frac{s}{\sqrt{s^2+t^2}}$$

$$\frac{\partial z}{\partial t} = e^r \cdot \cos\theta \cdot s + (-e^r \cdot \sin(\theta))\,\frac{t}{\sqrt{s^2+t^2}}$$

29. Use equation 6 to find dy/dx for $\tan^{-1}(x^2y) = x + xy^2$

$\boxed{\text{Equation 6}}$: $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$

Put equation on the form: $F(x,y) = 0 = \tan^{-1}(x^2y) - x - xy^2$

Take first partial derivatives: Remember: $\tan^{-1}(x)' = \dfrac{1}{1+x^2}$

$$F_x = \frac{1}{1+x^4y^2}\cdot 2xy - 1 - y^2 = \frac{2xy}{1+x^4y^2} - y^2 - 1 = \frac{2xy - (y^2+1)(1+x^4y^2)}{1+x^4y^2}$$

$$F_y = \frac{1}{1+x^4y^2}\cdot x^2 - 2xy = \frac{x^2}{1+x^4y^2} - 2xy = \frac{x^2 - (2xy)(1+x^4y^2)}{1+x^4y^2}$$

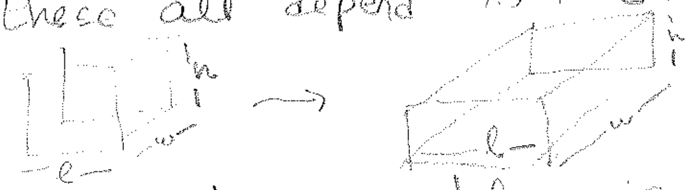
$$\frac{dy}{dx} = -\frac{\dfrac{2xy-(y^2+1)(1+x^4y^2)}{\cancel{1+x^4y^2}}}{\dfrac{x^2-(2xy)(1+x^4y^2)}{\cancel{1+x^4y^2}}} = -\frac{2xy-(y^2+1)(1+x^4y^2)}{x^2-(2xy)(1+x^4y^2)}$$

$$= -\frac{2xy - (y^2 + x^4y^4 + 1 + x^4y^2)}{x^2 - 2xy - 2x^5y^3} = \boxed{\frac{1 + x^4y^2 + y^2 + x^4y^4 - 2xy}{x^2 - 2xy - 2x^5y^3}}$$

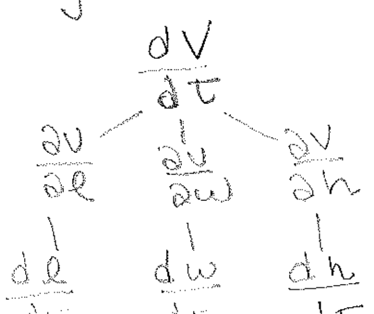
39. $V(l,w,h) = l\cdot w\cdot h$. At a given moment, say t_0, we have: $l = 1\ m$, $w = h = 2\ m$; l, w are increasing at a rate of $2\ m/s$ while h is decreasing at a rate of $3\ m/s$.

Hence, $l(t)$; $w(t)$; $h(t)$; these all depend on time!

$\dfrac{dl}{dt} = \dfrac{dw}{dt} = 2$; $\dfrac{dh}{dt} = -3$.



By CHAIN RULE.



the instantenous change in Volume is:

$$\frac{dV}{dt} = \frac{\partial V}{\partial l}\cdot\frac{dl}{dt} + \frac{\partial V}{\partial w}\cdot\frac{dw}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}.$$

At the given moment t_0

$$\frac{dV}{dt}(t_0) = w\cdot h\cdot 2 + l\cdot h\cdot 2 + lw\cdot(-3)$$
$$= 2^3 + 1\cdot 2^2 - 1\cdot 2\cdot 3 = 8 + 4 - 6 = +6\ m^3/s$$

Section 14.6

7. $f(x,y) = \sin(2x+3y)$, $P(-6,4)$, $\vec{u} = \frac{1}{2}(\sqrt{3}\,\hat{i} - \hat{j}) = \frac{1}{2}\langle \sqrt{3}, -1 \rangle$

(a) Find the gradient of f.

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2\cos(2x+3y), 3\cos(2x+3y) \rangle$$

(b) Evaluate the gradient at the point P.

$$\nabla f(-6,4) = \langle f_x(-6,4), f_y(-6,4) \rangle = \langle 2\cos(12-12), 3\cos(12-12) \rangle$$
$$= \langle 2\cos(0), 3\cos(0) \rangle = \boxed{\langle 2,3 \rangle}$$

(c) Find the rate of change of f at P in the direction of the vector $\vec{u}$

First, is $\vec{u}$ a unit vector?

$$|\vec{u}| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{-1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{\frac{4}{4}} = 1.$$ It is a unit vector.

Now,

$$\nabla f(-6,4) \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{-1}{2} \right\rangle = \langle 2,3 \rangle \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{-1}{2} \right\rangle = \sqrt{3} - \frac{3}{2} = \frac{2\sqrt{3}-3}{2}$$

---

10. $f(x,y,z) = y^2 e^{xyz}$, $P(0,1,-1)$ $\vec{u} = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle$

(a) Find the gradient of f

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle = \langle y^3 z e^{xyz}, 2y e^{xyz} + y^2 xz e^{xyz}, xy^3 e^{xyz} \rangle$$

(b) Evaluate the gradient at the point P

$$\nabla f(0,1,-1) = \langle 1^3(-1)\cdot e^{0(1)(-1)}, 2(1)\cdot e^{0(1)(-1)} + 1^2\cdot 0(-1)\cdot e^{0(1)(-1)}, 0\cdot 1^3\cdot e^{0(1)(-1)} \rangle$$
$$= \langle -1, 2, 0 \rangle$$

(c) Find the rate of change of f at P in the direction of $\vec{u}$

First, check that $\vec{u}$ is a unit vector:

$$|\vec{u}| = \sqrt{\left(\frac{3}{13}\right)^2 + \left(\frac{4}{13}\right)^2 + \left(\frac{12}{13}\right)^2} = \sqrt{\frac{9+16+144}{169}} = \sqrt{\frac{169}{169}} = 1.$$ $\vec{u}$ is a unit vector.

the rate of change is given by

$$\nabla f(0,1,-1) \cdot \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle = \langle -1,2,0 \rangle \cdot \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle$$
$$= -\frac{3}{13} + \frac{8}{13} = \boxed{\frac{5}{13}}$$

11. Find the directional derivative of the function at the given point in the direction of the vector $\vec{v}$.

$f(x,y) = e^x \sin(y)$, $(0, \pi/3)$, $\vec{v} = \langle -6, 8 \rangle$.

First, compute the gradient

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \langle e^x \sin(y), e^x \cos(y) \rangle$$

At the given point we have:

$$\nabla f(0, \pi/3) = \langle \sin(\pi/3), \cos(\pi/3) \rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle = \frac{1}{2} \langle \sqrt{3}, 1 \rangle$$

To compute the directional derivative we need a unit vector in the direction of $\vec{v}$:

$$|\vec{v}| = \sqrt{36+64} = \sqrt{100} = 10.$$

Hence, $$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -6, 8 \rangle}{10} = \left\langle -\frac{6}{10}, \frac{8}{10} \right\rangle = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$$

the directional derivative is:

$$\nabla f(0, \pi/3) \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle = \frac{1}{2} \langle \sqrt{3}, 1 \rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle =$$

$$= \frac{-3\sqrt{3}}{10} + \frac{4}{10} = \boxed{\frac{4-3\sqrt{3}}{10}}$$

---

13. Find the directional derivative of the function at the given point in the direction of the vector $\vec{v}$.

$g(p,q) = p^4 - p^2 q^3$, $(2,1)$, $\vec{v} = \hat{i} + 3\hat{j} = \langle 1, 3 \rangle$

First, compute the gradient:

$$\nabla g(p,q) = \langle g_p, g_q \rangle = \langle 4p^3 - 2pq^3, -3p^2q^2 \rangle$$

At the given point we have:

$$\nabla g(2,1) = \langle 4(2)^3 - 2(2)(1)^3, -3(2)^2(1)^2 \rangle = \langle 32-4, -12 \rangle = \langle 28, -12 \rangle$$

We need a unit vector

$|\vec{v}| = \sqrt{1+9} = \sqrt{10}$ $\Rightarrow$ the vector $\hat{v} = \frac{\langle 1, 3 \rangle}{\sqrt{10}} = \frac{1}{\sqrt{10}} \langle 1, 3 \rangle$

Hence, $\nabla g(2,1) \cdot \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle = \langle 28, -12 \rangle \cdot \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle = \frac{28}{\sqrt{10}} - \frac{36}{\sqrt{10}} = \boxed{\frac{-8}{\sqrt{10}}}$

25. Find the maximum rate of change of f at the given point and the direction in which it occurs.

$f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $(3,6,-2)$

the maximum rate of change occurs at the gradient:

$$\nabla f(x,y,z) = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

At the given point:

$$\nabla f(3,6,-2) = \left\langle \frac{3}{7}, \frac{6}{7}, \frac{-2}{7} \right\rangle = \frac{1}{7}\langle 3,6,-2\rangle$$

Hence, the maximum rate of change is:

$$|\nabla f(3,6,-2)| = \sqrt{\frac{9}{49}+\frac{36}{49}+\frac{4}{49}} = \sqrt{\frac{49}{49}} = 1$$

In the direction $\frac{1}{7}\langle 3,6,-2\rangle$

45. Find eqs. of (a) tangent plane and (b) normal line

$x+y+z = e^{xyz}$, $(0,0,1)$

First, compute the gradient of $f(x,y,z)$ defined as:

$x+y+z-e^{xyz}=0$, $\Rightarrow \nabla f(x,y,z) = \langle 1-yze^{xyz}, 1-xze^{xyz}, 1-xye^{xyz}\rangle$

At the given point:

$\nabla f(0,0,1) = \langle 1,1,1\rangle$ this is the normal vector.

For line:

$$\vec{\ell}(t) = (0,0,1) + t(1,1,1) = \langle \underset{x}{t}, \underset{y}{t}, \underset{z}{1+t}\rangle$$

For plane:

$$\nabla f(0,0,1)\cdot\langle x,y,z-1\rangle = 0 \Leftrightarrow \langle 1,1,1\rangle\cdot\langle x,y,z-1\rangle = 0$$

$$\boxed{x+y+z-1=0}$$

Section 14.6

(12) $f(x,y) = \dfrac{x}{x^2+y^2}$, $(1,2)$, $\vec{v} = \langle 3,5 \rangle$

First, compute the first partial derivative:

$$f_x = \frac{1}{x^2+y^2} + \frac{x(2x)}{(-1)(x^2+y^2)^2} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \boxed{\frac{y^2-x^2}{(x^2+y^2)^2}}$$

Using Quotient rule.

$$f_x = \frac{(1)\cdot(x^2+y^2) - (x)(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \boxed{\frac{y^2-x^2}{(x^2+y^2)^2}}$$

$$f_y = \frac{(x)\cdot(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

Hence, the gradient is: $\nabla f(x,y) = \left\langle \dfrac{y^2-x^2}{(x^2+y^2)^2}, \dfrac{-2xy}{(x^2+y^2)^2} \right\rangle$

the unit vector $\hat{v}$ in the direction of $\vec{v}$ is:

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 3,5 \rangle}{\sqrt{9+25}} = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$$

the directional derivative is

$$\nabla f(1,2) \cdot \hat{v} = \left\langle \frac{4-1}{5^2}, \frac{-2(1)(2)}{5^2} \right\rangle \cdot \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$$

$$= \left\langle \frac{3}{25}, \frac{-4}{25} \right\rangle \cdot \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle = \frac{9}{25\sqrt{34}} - \frac{20}{25\sqrt{34}} = \boxed{\frac{-11}{25\sqrt{34}}}$$

(22) $f(s,t) = t e^{st}$, $(0,2)$

$\nabla f(s,t) = \langle f_s, f_t \rangle = \langle t^2 e^{st}, e^{st} + st e^{st} \rangle$

the maximum rate of change of at the point $(0,2)$ is

$|\nabla f(0,2)| = \sqrt{(4)^2 + (1)^2} = \sqrt{16+1} = \boxed{\sqrt{17}}$

the direction is given by

$\nabla f(0,2) = \boxed{\langle 4, 1 \rangle}$

---

(42) $y = x^2 - z^2$, $(4,7,3)$

$F(x,y,z) = x^2 - y \quad z^2$

$\nabla F(x,y,z) = \langle 2x, -1, -2z \rangle$

the plane is given by:

$\nabla F(4,7,3) \cdot \langle x-4, y-7, z-3 \rangle = 0$

$\langle 8, -1, -6 \rangle \cdot \langle x-4, y-7, z-3 \rangle = 0$

$8x - 32 - y + 7 \quad -6z + 18 = 0$

$\boxed{\begin{array}{l} 8x - y - 6z + 7 = 0 \\ 8x - y - 6z = 7 \end{array}}$ divide both sides by

the normal line is given by:

$\boxed{\vec{l}(s) = \langle 4, 7, 3 \rangle + s \langle 8, -1, -6 \rangle}$

$\vec{l}(s) = \langle 4+8s, 7-s, 3-6s \rangle$

$x(s) = 4 + 8s$; $y(s) = 7 - s$; $z(s) = 3 - 6s$

$\frac{x-4}{\ } = \frac{y-7}{\ } = \frac{z-3}{\ }$

(44) $xy + yz + zx = 5$ $(1,2,1)$

$F(x,y,z) = xy + yz + zx - 5 = 0$

$\nabla F(x,y,z) = \langle y+z, x+z, y+x \rangle$

the plane is given by:

$\nabla F(1,2,1) \cdot \langle x-1, y-2, z-1 \rangle = 0$

$\langle 3,2,3 \rangle \cdot \langle x-1, y-2, z-1 \rangle = 0$

$3x - 3 + 2y - 4 + 3z - 3 = 0$

$3x + 2y + 3z - 10 = 0$

$\boxed{3x + 2y + 3z = 10}$

the line is given by:

$\vec{l}(s) = \langle 1,2,1 \rangle + s\langle 3,2,3 \rangle$

$x(s) = 1 + 3s$; $y(s) = 2 + 2s$; $z(s) = 1 + 3s$

$$\boxed{\frac{x-1}{3} = \frac{y-2}{2} = \frac{z-1}{3}}$$

$g_s(1,2) = f_x(0,0) \cdot X_s(1,2) + f_y(0,0) \cdot Y_s(1,2)$.

$X_s = -1$; $y_s = 2 \cdot s \Rightarrow y_s(1,2) = 4$

$g_s(1,2) = 4 \cdot (-1) + 8 \cdot 4 = -4 + 32 = \boxed{28}$

---

(30) $e^y \sin(x) = x + xy \Rightarrow e^y \sin(x) - x - xy = 0 = F(x,y)$.

$\frac{dy}{dx} = -\frac{F_x}{F_y}$; $F_x = e^y \cos(x) - 1 - y$

$F_y = e^y \cdot \sin(x) - x$

$\Rightarrow \frac{dy}{dx} = -\left(\frac{e^y \cdot \cos(x) - 1 - y}{e^y \cdot \sin(x) - x}\right) = \boxed{\frac{y + 1 - e^y \cdot \cos(x)}{e^y \cdot \sin(x) - x}}$

---

(32) $x^2 - y^2 + z^2 - 2z = 4 \Rightarrow F(x,y,z) = x^2 - y^2 + z^2 - 2z - 4$

$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \boxed{= -\frac{2x}{2z-2}}$; $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{-2y}{2z-2} = \boxed{\frac{2y}{2z-2}}$

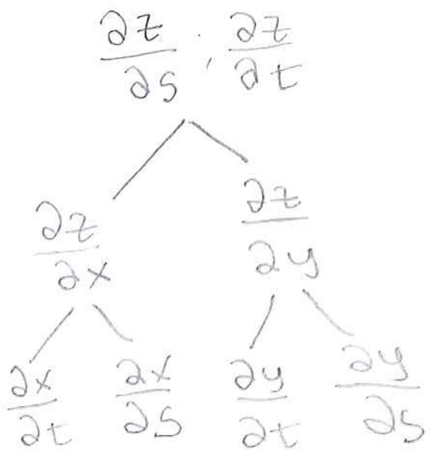
---

(34) $yz + x\ln(y) = z^2 \Rightarrow F(x,y,z) = yz + x\ln(y) - z^2$

$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = \boxed{\frac{-\ln(y)}{y - 2z}}$; $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = \frac{z + \frac{x}{y}}{y - 2z} = \boxed{-\frac{\frac{zy + x}{y}}{y - 2z}}$

Section 14.5

10) $z = e^{x+2y}$; $x = s/t$, $y = t/s$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial x} = e^{x+2y} \; ; \; \frac{\partial z}{\partial y} = 2e^{x+2y}$$

$$\frac{\partial x}{\partial s} = \frac{1}{t} \; ; \; \frac{\partial x}{\partial t} = -\frac{s}{t^2} \; ; \; \frac{\partial y}{\partial s} = -\frac{t}{s^2} \; ; \; \frac{\partial y}{\partial t} = \frac{1}{s}$$

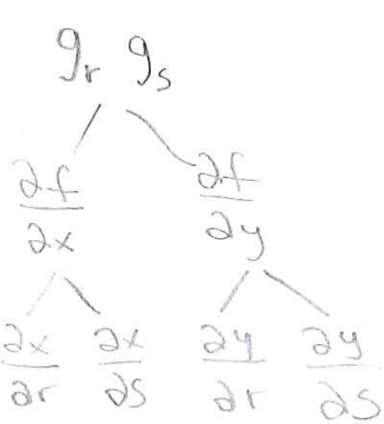
Hence,

$$\boxed{\frac{\partial z}{\partial s} = (e^{x+2y}) \cdot \frac{1}{t} + (2e^{x+2y})\left(-\frac{t}{s^2}\right)} \quad //$$

$$\boxed{\frac{\partial z}{\partial t} = -(e^{x+2y}) \cdot \frac{s}{t^2} + (2e^{x+2y}) \cdot \frac{1}{s}} \quad //$$

---

16) $g(r,s) = f(\overset{x}{\overset{\shortparallel}{2r-s}}, \overset{y}{\overset{\shortparallel}{s^2-4r}})$



$$g_r = \frac{\partial g}{\partial r} = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}$$

$$g_s = \frac{\partial g}{\partial s} = \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

We want to find

$g_r(1,2) \Rightarrow (r,s) = (1,2)$

$\Rightarrow (x,y) = (0,0)$.

$$g_r(1,2) = f_x(0,0) \cdot x_r(1,2) + f_y(0,0) \cdot y_r(1,2).$$

From the table: $f_x(0,0) = 4$; $f_y(0,0) = 8$.

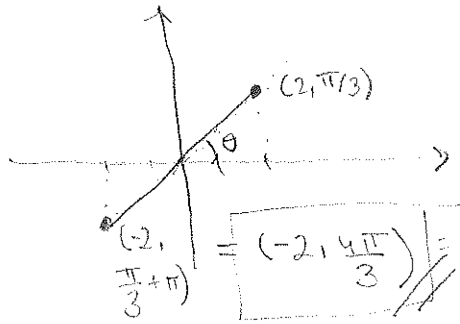
New, $x_r = 2$; $y_r = -4$. $\Rightarrow g_r(1,2) = 4 \cdot 2 - 4 \cdot 8 = \boxed{-24}$ //

Section 10.3

1 (a) $(2, \pi/3)$

$= (2, \pi/3 + 2\pi)$

$= \boxed{\left(2, \frac{7\pi}{3}\right)}$

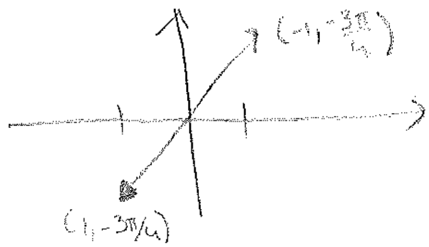


$= \boxed{\left(-2, \frac{4\pi}{3}\right)}$

(b) $(1, -3\pi/4)$

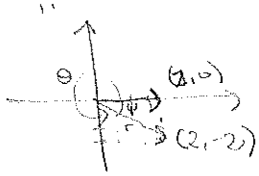
$\left(1, -\frac{3\pi}{4} + 2\pi\right) =$

$\left(1, \frac{5\pi}{4}\right)$



$\left(-1, -\frac{3\pi}{4} + \pi\right) = \left(-1, \frac{\pi}{4}\right)$

5 (a) $(2,-2)$



$(2,0)\cdot(2,-2) = 4 = |(2,0)||(2,-2)|\cos\psi$
$= 1\,4\sqrt{2}\cdot\cos\psi$
$\Rightarrow \psi = \arccos\left(\frac{1}{\sqrt{2}}\right) = \arccos\left(\frac{\sqrt{2}}{2}\right)$
$\Rightarrow \psi = 45^\circ = \frac{\pi}{4}$

$\cos\psi = \frac{|\langle 2,0\rangle|}{r}$
$\Rightarrow \cos 45^\circ = \frac{2}{r} \Rightarrow r = \frac{2}{\frac{1}{\sqrt{2}}} = 2\sqrt{2}$

Since $\theta^\circ + \psi = 360^\circ$

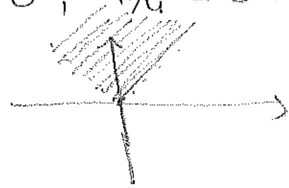
Hence, $\theta^\circ = 360^\circ - \varphi = 360^\circ - 45^\circ = 315^\circ = \frac{7\pi}{4}$

$(2,-2)$ is equivalent to $\left(2\sqrt{2}, \frac{7\pi}{4}\right)$

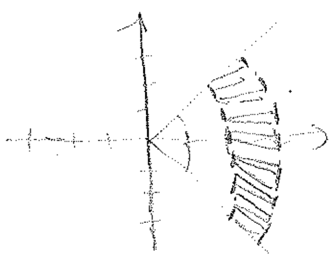
which is equivalent to $\left(-2\sqrt{2}, \frac{7\pi}{4} + \pi\right) = \left(-2\sqrt{2}, \frac{11\pi}{4}\right)$

(7) $r \geq 1$

(9) $r \geq 0$, $\pi/4 \leq \theta \leq 3\pi/4$



(11) $2 < r < 3$, $5\pi/3 \leq \theta \leq 7\pi/3$



(15) $r^2 = 5$

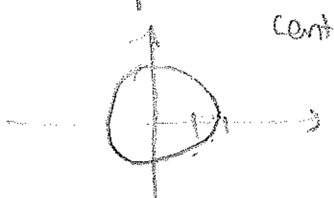
$r = \pm\sqrt{5}$

$x = r\cdot\cos\theta$; $y = r\cdot\sin\theta$

Circle of radius $\sqrt{5}$ center 0.

$r^2 = x^2 + y^2 = 5$

$\Rightarrow x^2 + y^2 = 5$



(17) $r = 2\cos\theta$ $\quad x = r.\cos\theta \quad ; \quad y = r.\sin\theta \Rightarrow x = 2\cos^2\theta$
$y = 2.\cos\theta.\sin\theta$

$r^2 = 4\cos^2\theta \qquad r^2 = x^2 + y^2$

$\Rightarrow \qquad \cos\theta = \frac{x}{r}$

$r = \frac{2x}{r} \Rightarrow r^2 = 2x = x^2 + y^2 \Rightarrow x^2 - 2x + y^2 = 0$

$\Rightarrow (x-1)^2 + y^2 - 1 = 0$

Circle of radius 1 centered at $(1,0)$.

$\boxed{(x-1)^2 + y^2 = 1}$

(19) $r^2\cos 2\theta = 1 \quad ; \quad x = r.\cos\theta \quad ; \quad y = r.\sin\theta \quad ; \quad r^2 = x^2 + y^2$

$\Leftrightarrow r^2(\cos^2 x - \sin^2 x) = 1$

$\Leftrightarrow r^2\cos^2 x - r^2\sin^2 x = 1$

$\boxed{x^2 - y^2 = 1}$

Hyperbola, center θ, foci on x-axis

(21) $y = 2 \quad ; \quad x = r.\cos\theta \quad ; \quad y = r.\sin\theta \quad ; \quad r^2 = x^2 + y^2$

$\boxed{2 = r.\sin\theta}$

$r = \frac{2}{\sin\theta}$

$\boxed{r = 2\csc(\theta)}$

$r^2 = r^2.\cos^2\theta + 2$

$r^2(1 - \cos\theta) = 2$

$\boxed{r^2 = \frac{2}{1-\cos^2\theta}}$

(23) $y = 1 + 3x \quad ; \quad x = r.\cos\theta \quad ; \quad y = r.\sin\theta \quad ; \quad r^2 = x^2 + y^2$

$r.\sin\theta = 1 + 3(r.\cos\theta)$

$r\sin\theta = 1 + 3r\cos\theta$

$r\sin\theta - 3r\cos\theta = 1$

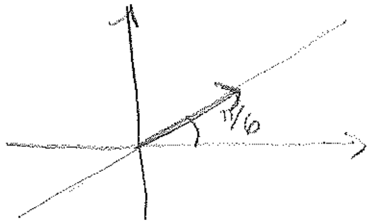
$\boxed{r = \frac{1}{\sin\theta - 3\cos\theta}}$

(25) $x^2 + y^2 = 2cx \quad ; \quad x = r\cos\theta \quad ; \quad y = r\sin\theta \quad ; \quad r^2 = x^2 + y^2$

$r^2 = 2cx$

$r^2 = 2c(r\cos\theta) \Rightarrow \boxed{r = 2c\cos(\theta)}$

27 (a) A line through the origin that makes an angle of $\pi/6$ with the positive x-axis



In polar coordinates $\boxed{\text{EASIER}}$

$\boxed{\theta = \pi/6}$

In Cartesian: the unit direction vector is $\vec{v}$ s.t.

$\vec{v} \cdot \langle 1,0 \rangle = 1 \cdot 1 \cdot \cos\theta$

$\langle v_1, v_2 \rangle \cdot \langle 1,0 \rangle = \cos\theta \Rightarrow v_1 = \cos(\pi/6)$

$\Rightarrow v_1 = \frac{\sqrt{3}}{2}$

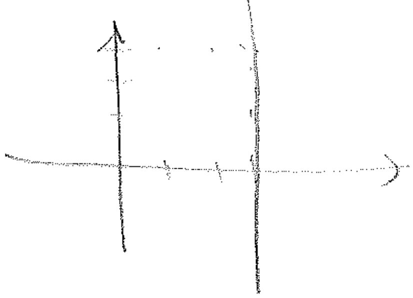
Hence, $v_2 = \frac{1}{2} \Rightarrow \vec{v} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$ $|\vec{v}| = 1$.

the line is $\vec{l}(t) = \langle 0,0 \rangle + t \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle = \langle \frac{\sqrt{3}}{2} t, \frac{1}{2} t \rangle$

$x(t) = \frac{\sqrt{3}}{2} t \quad y(t) = \frac{1}{2} t \Rightarrow \frac{x}{\sqrt{3}} \cdot \cancel{2} = t = y \cdot \cancel{2}$

$\Rightarrow \boxed{y = \frac{x}{\sqrt{3}}}$

(b) A vertical line through the point $(3,3)$



In Cartesian coordinates: $\boxed{\text{EASIER}}$

$x = 3$

In polar coordinates

$x = r\cos\theta = 3 \Rightarrow \boxed{r = \frac{3}{\cos\theta}}$

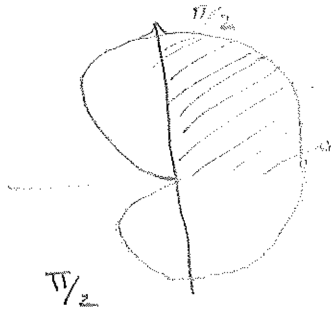
(5) Find the area of $r = \sqrt{\theta}$, $a = 0 \leq \theta \leq 2\pi = b$

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} (\sqrt{\theta})^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} \theta \, d\theta = \frac{1}{2} \left[\frac{\theta^2}{2} \right]_0^{2\pi}$$

$$= \frac{(2\pi)^2}{4} = \frac{\cancel{4}\pi^2}{\cancel{4}} = \boxed{\pi^2}$$

Similar to Section 10.4 (5-8)

Find the area of the shaded region



$r = 1 + \cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1$
$\Leftrightarrow \cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$

$$\text{Area} = \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} (1+\cos\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} 1 + 2\cos\theta + \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\left[\int_0^{\pi/2} 1 \, d\theta + 2\int_0^{\pi/2} \cos\theta \, d\theta + \int_0^{\pi/2} \cos^2\theta \, d\theta\right]$$

$$= \frac{1}{2}\left[[\theta]_0^{\pi/2} + 2[\sin\theta]_0^{\pi/2} + \int_0^{\pi/2} \frac{1+\cos 2\theta}{2} \, d\theta\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + 2 + \frac{1}{2}\left[\int_0^{\pi/2} 1 \, d\theta + \int_0^{\pi/2} \cos 2\theta \, d\theta\right]\right]$$

$$= \frac{1}{2}\left[\pi/2 + \frac{1}{2}\left[(\theta)_0^{\pi/2} + \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}\right]\right]$$

$$= \frac{1}{2}\left[\pi/2 + \frac{1}{2}\left[\frac{\pi}{2} + 0\right]\right]$$

$$= \frac{1}{2}\left[\pi/2 + \frac{\pi}{4}\right]$$

$$= \frac{1}{2}\left[\frac{2\pi + \pi}{4}\right]$$

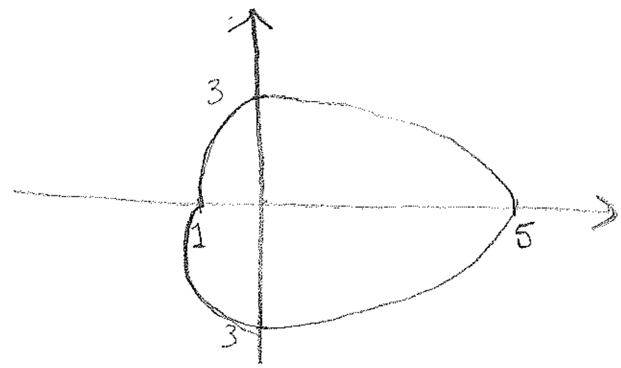
$$= \frac{1}{2}\left[\frac{3\pi}{4}\right]$$

$$= \boxed{\frac{3\pi}{8}}$$

(11) Sketch the curve and find the area that it encloses.

$r = 3 + 2\cos\theta$

r	θ
5	0
3	$\pi/2$
1	π
3	$3\pi/2$



$$\text{Area} = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} (3+2\cos\theta)^2 \, d\theta$$

$$= \dots = \boxed{11\pi}$$

(23)

Find the area inside $r = 2\cos\theta$ but outside $r = 1$.

First, graph the curves:

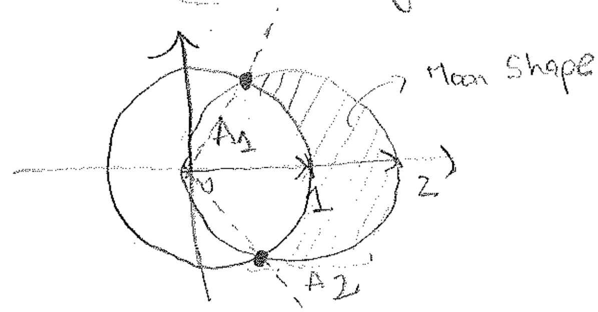
$r = 2\cos\theta \iff \cos\theta = \frac{r}{2}$; $\cos\theta = \frac{x}{r} \Rightarrow \frac{r}{2} = \frac{x}{r}$

$\Rightarrow r^2 = 2x = x^2 + y^2 \Rightarrow x^2 + y^2 - 2x = 0 \equiv (x-1)^2 + y^2 = 1$

Unit circle centered at $(1,0)$.

$r = 1 \Rightarrow r^2 = x^2 + y^2 = 1$

Unit circle centered at $(0,0)$



Second, find the points of intersection:

$$r = 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pi/3 \text{ or } \theta = -\pi/3$$

Finally, compute the area by substracting A2 from A1:

$$\text{Area} = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta)^2 - (1)^2 \, d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4\cos^2\theta - 1 \, d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 2\cos(2\theta) + 2 - 1 \, d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} 2\cos(2\theta) + 1 \, d\theta$$

$$= \frac{1}{2}\left[\left[\frac{2}{2}\sin(2\theta)\right]_{-\pi/3}^{\pi/3} + \left[\theta\right]_{-\pi/3}^{\pi/3}\right] = \frac{1}{2}\left[\left[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right] + \left[\frac{\pi}{3} + \frac{\pi}{3}\right]\right]$$

$$= \frac{1}{2}\left[\sqrt{3} + \frac{2\pi}{3}\right] = \boxed{\frac{\sqrt{3}}{2} + \frac{\pi}{3}}$$

Similar to 45-48

Find the exact length of the polar curve

$$r = 5\cos\theta \quad ; \quad 0 \le \theta \le \frac{3\pi}{4}$$

the length is given by:

$$L = \int_0^{3\pi/4} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{where} \quad \frac{dr}{d\theta} = -5\sin\theta \text{ , hence:}$$

$$L = \int_0^{3\pi/4} \sqrt{(5\cos\theta)^2 + (-5\sin\theta)^2}\, d\theta$$

$$= \int_0^{3\pi/4} \sqrt{25(\sin^2\theta + \cos^2\theta)}\, d\theta$$

$$= \int_0^{3\pi/4} \sqrt{25}\, d\theta$$

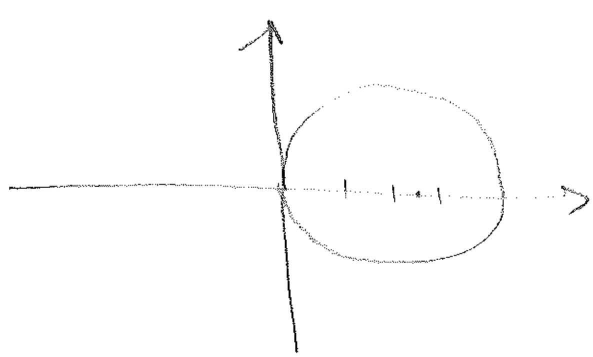
$$= \int_0^{3\pi/4} 5\, d\theta = 5\,[\theta]_0^{3\pi/4} = 5 \cdot \frac{3\pi}{4} = \boxed{\frac{15\pi}{4}}$$

Graphing the curve r:

$$x = r\cos\theta \Rightarrow r = \frac{x}{\cos\theta} \quad ; \quad \cos\theta = \frac{r}{5} \quad \Rightarrow \quad r = \frac{x}{\frac{r}{5}} = \frac{5x}{r} \Rightarrow r^2 = 5x$$

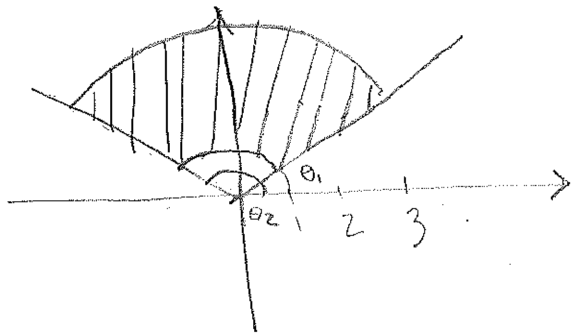
$$r^2 = 5x = x^2 + y^2 \Rightarrow x^2 - 5x + y^2 = 0 \Leftrightarrow \left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$$

circle centered at $\left(\frac{5}{2}, 0\right)$ with radius $\frac{5}{2}$



SECTION 10.3

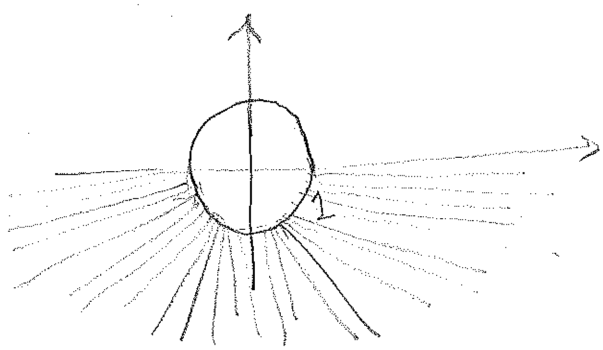
(10) $1 \leq r \leq 3$, $\pi/6 < \theta < 5\pi/6$



$\theta_1 = \pi/6$

$\theta_2 = 5\pi/6$

(12) $r \geq 1$, $\pi \leq \theta \leq 2\pi$



outside unit circle, including circumference, on the 3rd and 4th quadrant

(16) $r = 4\sec\theta = \frac{4}{\cos(\theta)}$; $\cos(\theta) = \frac{4}{r}$ $\quad \cos\theta = \frac{x}{r}$

$\frac{4}{r} = \frac{x}{r} \Rightarrow \boxed{x = 4}$

(22) $y = x \Rightarrow \boxed{\theta = \frac{\pi}{4}}$

(24) $4y^2 = x$ $\quad x = r.\cos\theta = 4y^2 = 4(r\cdot\sin\theta)^2 = 4\cdot r^2\sin^2\theta$

$y = r.\sin\theta$ $\quad \Rightarrow \cancel{r}\cos\theta = 4\cdot r^{\cancel{2}}\sin^2\theta$

$\boxed{r = \frac{\cos\theta}{4\sin^2\theta}}$

SECTION 10.4

(8)  $r = \sin 2\theta$ $\quad \frac{1}{2}\int_0^{\pi/2} (\sin 2\theta)^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta)\, d\theta$

make the change : $2\theta = \varphi \quad 2d\theta = d\varphi \Rightarrow d\theta = \frac{d\varphi}{2}$

$$\Rightarrow \frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta)d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{\sin^2(\varphi)\, d\varphi}{2} = \frac{1}{4}\int_0^{\pi/2} \sin^2(\varphi)\, d\varphi$$

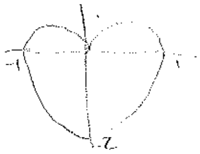
$$= \frac{1}{4}\left[\frac{1}{2}\varphi - \frac{1}{4}\sin(2\varphi)\right]$$ change varibles

$$= \frac{1}{4}\left[\frac{1}{2}(2\theta) - \frac{1}{4}\sin(4\theta)\right]_0^{\pi/2} = \frac{1}{2}\left[\frac{\theta}{2} - \frac{\sin(4\theta)}{8}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left[\frac{\pi}{4} - \cancel{\frac{\sin(2\pi)}{8}}\right] - \left[0 - 0\right]\right] = \frac{1}{2}\left(\frac{\pi}{4}\right) = \boxed{\frac{\pi}{8}}$$

---

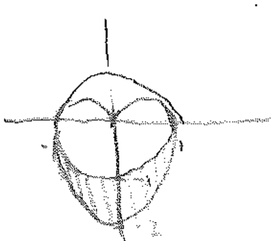
(10) $r = 1 - \sin\theta$ $\quad 2\cdot\frac{1}{2}\int_{-\pi/2}^{\pi/2} (1-\sin\theta)^2 d\theta = \int_{-\pi/2}^{\pi/2} 1 - 2\sin\theta + \sin^2\theta\, d\theta$



$$= [\theta]_{-\pi/2}^{\pi/2} - 2\left[-\cos\theta\right]_{-\pi/2}^{\pi/2} + \left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_{-\pi/2}^{\pi/2}$$

$$= \pi/2 + \pi/2 + \left[\left(\frac{\pi}{4} - 0\right) - \left(\frac{-\pi}{4} - 0\right)\right] = \pi + \left[\frac{2\pi}{4}\right] = \pi + \frac{\pi}{2} = \boxed{\frac{3\pi}{2}}$$

(24) $r = 1 - \sin\theta$ $\quad , r = 1$



$$\frac{1}{2}\int_{\pi}^{2\pi} (1-\sin\theta)^2 - (1)^2\, d\theta$$

$$A_1 = \frac{1}{2}\int_{\pi}^{2\pi} (1-\sin\theta)^2 d\theta$$ = previosly calculated

$$= \left[\theta + 2\cos\theta + \frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_{\pi}^{2\pi}$$

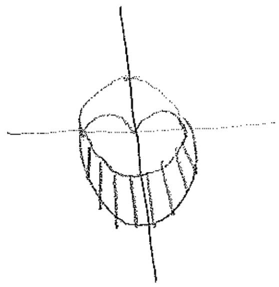
$$= \left[(2\pi + 2 + \pi - 0) - \left(\pi - 2 + \frac{\pi}{2} - 0\right)\right] = 3\pi + 2 - \frac{3\pi}{2} + 2 = \frac{3\pi}{2} + 4$$

the area of the semicircle is $\frac{\pi}{2}$

Hence, the shadded area is $\frac{3\pi}{2} + 4 - \frac{\pi}{2} = \pi + 4$

Section 10.4

(24) $r = 1 - \sin\theta$; $r = 1$



$$\frac{1}{2}\int_{\pi}^{2\pi} (1-\sin\theta)^2 - (1)^2 \, d\theta$$

$$= \frac{1}{2}\int_{\pi}^{2\pi} 1 - 2\sin\theta + \sin^2\theta - 1 \, d\theta$$

$$= \frac{1}{2}\int_{\pi}^{2\pi} \sin^2\theta - 2\sin\theta \, d\theta = \frac{1}{2}\left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + 2\cos(\theta)\right]_{\pi}^{2\pi}$$

$$\frac{1}{2}\left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4} + 2\cos(\theta)\right]_{\pi}^{2\pi} = \frac{1}{2}\left[\left(\frac{2\pi}{2} - 0 + 2\right) - \left(\frac{\pi}{2} - 0 - 2\right)\right]$$

$$= \frac{1}{2}\left[\frac{2\pi}{2} - \frac{\pi}{2} + 2 + 2\right] = \frac{1}{2}\left[\frac{\pi}{2} + 4\right] = \boxed{\frac{\pi}{4} + 2}$$

(28) $r = 3\sin\theta$, $r = 2 - \sin\theta$

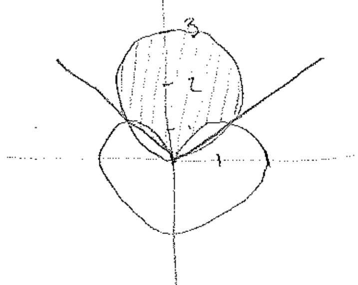
First, compute the points of intersection:

$r = 3\sin\theta = 2 - \sin\theta \Rightarrow 3\sin\theta - 2 + \sin\theta = 0$

$4\sin\theta - 2 = 0 \iff 2(2\sin\theta - 1) = 0$

$\Rightarrow 2\sin\theta - 1 = 0 \Rightarrow \sin\theta = \frac{1}{2}$

$\Rightarrow \theta = 45° = \boxed{\frac{\pi}{6}}$ or $\theta = \boxed{\frac{5\pi}{6}}$



$$\frac{1}{2}\int_{\pi}(3\sin\theta)^2 - (2-\sin(\theta))^2 \, d\theta = \frac{1}{2}\int 9\sin^2\theta - (4 - 4\sin\theta + \sin^2\theta) \, d\theta$$

$$= \frac{1}{2}\int 9\sin^2\theta - \sin^2\theta + 4\sin\theta - 4 \, d\theta = \frac{1}{2}\int 8\sin^2\theta + 4\sin\theta - 4 \, d\theta = 2\int 2\sin^2\theta + \sin\theta - 1 \, d\theta$$

$$= 2\left[2\left(\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right) - \cos\theta - \theta\right] = \left[2\theta - \sin(2\theta) - \cos\theta - \theta\right]$$

$$= \left[\theta - \sin(2\theta) - \cos(\theta)\right]_{\pi/6}^{5\pi/6} = \left(\frac{5\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)$$

$$= \frac{5\pi}{6} + \frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \boxed{\pi + 2\sqrt{3}}$$

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2 - (2-\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 9\sin^2\theta - (4 - 4\sin\theta + \sin^2\theta) \, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 9\sin^2\theta - \sin^2\theta + 4\sin\theta - 4 \, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 8\sin^2\theta + 4\sin\theta - 4 \, d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} 4(2\sin^2\theta + \sin\theta - 1) \, d\theta$$

$$= 2\int_{\pi/6}^{5\pi/6} 2\sin^2\theta + \sin\theta - 1 \, d\theta = \underbrace{\int_{\pi/6}^{5\pi/6} 4\sin^2\theta}_{A_1} + \underbrace{2\sin\theta}_{A_2} \underbrace{- 2}_{A_3} \, d\theta$$

$$A_1 = 4\int_{\pi/6}^{5\pi/6} \sin^2\theta = 4\left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_{\pi/6}^{5\pi/6} = \left[2\theta - \sin(2\theta)\right]_{\pi/6}^{5\pi/6}$$

$$= \left(\frac{5\pi}{3} - \sin\left(\frac{5\pi}{3}\right)\right) - \left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right) = \left(\frac{5\pi}{3} + \frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$= \frac{5\pi}{3} - \frac{\pi}{3} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \boxed{\frac{4\pi}{3} + \sqrt{3}} = A_1$$

$$A_2 = 2\int_{\pi/6}^{5\pi/6} \sin\theta \, d\theta = -2\left[\cos\theta\right]_{\pi/6}^{5\pi/6} = -2\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = -2(-\sqrt{3}) \boxed{= 2\sqrt{3}}$$

$$A_3 = 2\int_{\pi/6}^{5\pi/6} d\theta = 2\left[\theta\right]_{\pi/6}^{5\pi/6} = 2\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) = 2\left(\frac{4\pi}{6}\right) = 2\left(\frac{2\pi}{3}\right) = \boxed{\frac{4\pi}{3}}$$

$$A = A_1 + A_2 - A_3 = \cancel{\frac{4\pi}{3}} + \sqrt{3} + 2\sqrt{3} - \cancel{\frac{4\pi}{3}} = \boxed{3\sqrt{3}}$$

(48) Find the exact length of the polar curve

$r = 2(1+\cos\theta)$ $\quad \boxed{\frac{dr}{d\theta} = -2\sin\theta}$ $\quad 2u=\theta \Rightarrow u = \frac{\theta}{2}$

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$\cos(2u) = 2\cos^2(u) - 1$

$\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

$$= \int_0^{2\pi} \sqrt{(2+2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4+8\cos\theta+4}\, d\theta = \int_0^{2\pi} \sqrt{8+8\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{8(1+\cos\theta)}\, d\theta = \int_0^{2\pi} 2\sqrt{2+2\cos\theta} = 2\int_0^{2\pi} \sqrt{2+2\cos\theta}\, d\theta$$

$$= 2\int_0^{2\pi} \sqrt{2+2\left(2\cos^2\left(\frac{\theta}{2}\right)-1\right)}\, d\theta = 2\int_0^{2\pi} \sqrt{2+\left(4\cos^2\left(\frac{\theta}{2}\right)-2\right)}\, d\theta$$

$$= 2\int_0^{2\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\, d\theta = 2\int_0^{2\pi} 2\cos\left(\frac{\theta}{2}\right) d\theta = 4\int_0^{2\pi} \cos\left(\frac{\theta}{2}\right) d\theta$$

$$4\left[2\cdot\sin\left(\frac{\theta}{2}\right)\right]_0^{2\pi} = 8\left[\sin\left(\frac{\theta}{2}\right)\right]_0^{2\pi} = 8(-1-0)$$

$r = 2(1+\cos\theta) \Leftrightarrow$

$r = 2 + 2\cos\theta$

$r' = -2\sin\theta$

$\cos^2 u = \dfrac{1+\cos(2u)}{2}$

$2\cdot\cos^2 u - 1 = \cos(2u)$

If $2u = \theta \Rightarrow u = \dfrac{\theta}{2}$

$$L = \int_0^{2\pi} \sqrt{(2+2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{8 + 8\left(2\cdot\cos^2\left(\frac{\theta}{2}\right) - 1\right)}\, d\theta = \int_0^{2\pi} \sqrt{8 + 16\cdot\cos^2\left(\frac{\theta}{2}\right) - 8}$$

$$= \int_0^{2\pi} \sqrt{16\cos^2\left(\frac{\theta}{2}\right)} = \int_0^{2\pi} 4\cos\left(\frac{\theta}{2}\right) = 4\int_0^{2\pi} \left|\cos\left(\frac{\theta}{2}\right)\right| d\theta$$

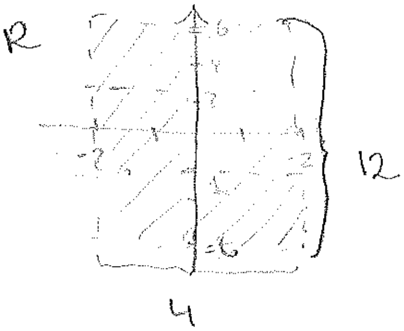
$$= 4\left[\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta - \int_{\pi}^{2\pi} \cos\left(\frac{\theta}{2}\right) d\theta\right]$$

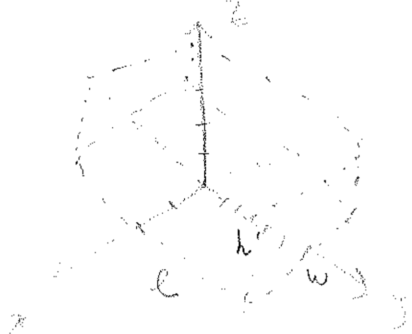
$$= 4\left[\left[2\cdot\sin\left(\frac{\theta}{2}\right)\right]_0^{\pi} - \left[2\cdot\sin\left(\frac{\theta}{2}\right)\right]_{\pi}^{2\pi}\right]$$

$$= 4\left[[2-0] - [0-2]\right] = 4[4] = \boxed{16}$$

Section 15.1

Similar to (11) $\iint_R 3\,dA$, $R=\{(x,y) \mid -2\le x\le 2, -6\le y\le 6\}$





Parallelepiped with

Volume is $V = l\cdot h\cdot w$,

where $l=12;\ w=4;\ h=3$

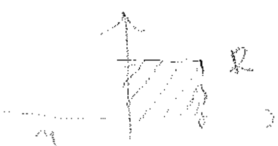
$V = 12\times 4\times 3 = 12\times 12 = \boxed{144}$

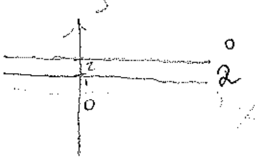
---

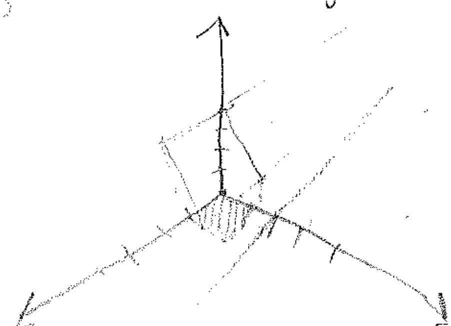
(13) $\iint_R (4-2y)\,dA$, $R=[0,1]\times[0,1]$

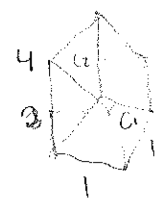
$z = 4-2y$, fix $z = k$ a constant

$k = 4-2y \Rightarrow \frac{k-4}{-2} = y \Rightarrow y = 2 - \frac{k}{2}$









C_1 = Cube of side S

$l=1$ $w=1$ $h=2$

$V_{C_1} = 2\cdot 1\cdot 1 = 2$

$C_2 = \frac{1}{2}$ cube C_1

$V_{C_2} = \frac{1}{2}\cdot 2 = 1$

Area $= V_{C_1} + V_{C_2} = 2+1 = \boxed{3}$

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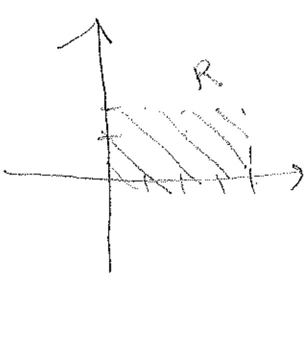
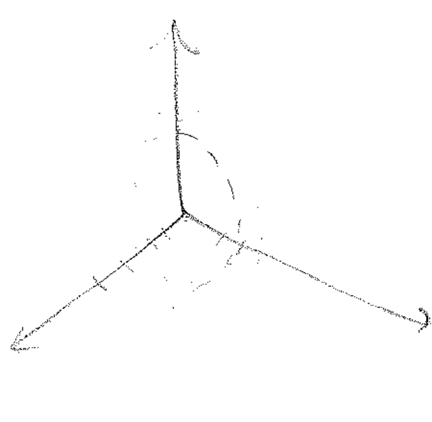
(14) $\iint_R \sqrt{9-y^2}\,dA$, $R=[0,4]\times[0,2]$

$z=\sqrt{9-y^2} \iff z^2 = 9-y^2 \iff z^2+y^2 = 9 \iff z^2+y^2 = 3^2$





(17) If f is a constant function $f(x,y)=k$, and $R=[a,b]\times[c,d]$,
Show that
$$\iint_R k\,dA = k(b-a)(d-c)$$

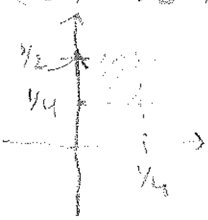
<u>Solution:</u>

$$\iint_R k\,dA = k\iint_R dA = k\iint_R dy\,dx = k\int_a^b\int_c^d dy\,dx$$

$$= k\int_a^b [y]_c^d\,dx = k\int_a^b (d-c)\,dx = k(d-c)\int_a^b dx = k(d-c)[x]_a^b$$

$$= \boxed{k(b-a)(d-c)}$$

(18) Use the result of Exercise 17 to show that:



$$0 \le \iint_R \sin \pi x \cos \pi y\,dA \le \frac{1}{32} \quad \text{where } R=[0,\tfrac{1}{4}]\times[\tfrac{1}{4},\tfrac{1}{2}]$$

x ranges from 0 to 1/4
y ranges from 1/4 to 1/2, hence,

If $x=0$ and $y=1/4$

$$\iint_R \sin(\pi x)\cos(\pi y)\,dA = \iint_R \sin(0)\cos(\tfrac{\pi}{4})\,dA = \iint_R 0\,dA = 0$$

If $x=\frac{1}{4}$ and $y=1/2$

$$\iint_R \sin(\tfrac{\pi}{4})\cos(\tfrac{\pi}{2})\,dA = \iint_R \sin(\tfrac{\pi}{4})\,0\,dA = \iint_R 0\,dA = 0$$

If $x=\frac{1}{4}$ and $y=\frac{1}{4}$

$$\iint_R \sin(\tfrac{\pi}{4})\cos(\tfrac{\pi}{4})\,dA = \iint_R \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\,dA = \iint_R \frac{2}{4}\,dA = \frac{1}{2}\iint_R dA$$

By (17) $= \frac{1}{2}(\frac{1}{4}-0)(\frac{1}{2}-\frac{1}{4}) = \frac{1}{2}(\frac{1}{4})(\frac{1}{4}) = \boxed{\frac{1}{32}}$

SECTION 15.2

(5) Calculate the iterated integral

$$\int_0^2\int_0^4 y^3 e^{2x}\,dy\,dx = \int_0^2 e^{2x}\left[\int_0^4 y^3\,dy\right]dx = \int_0^2 e^{2x}\left[\frac{y^4}{4}\right]_0^4 dx$$

$$= \int_0^2 e^{2x}\left(\frac{4^4}{4}\right)dx = 64\int_0^2 e^{2x}\,dx = 64\left[\frac{e^{2x}}{2}\right]_0^2 = 32\left[e^4 - e^0\right] = \boxed{32\left[e^4-1\right]}$$

(9) $$\int_1^4\int_1^2\left(\frac{x}{y}+\frac{y}{x}\right)dy\,dx = \int_1^4\left[\int_1^2 \frac{x}{y}\,dy + \int_1^2 \frac{y}{x}\,dy\right]dx$$

$$= \int_1^4\left[x\int_1^2 \frac{1}{y}\,dy + \frac{1}{x}\int_1^2 y\,dy\right]dx = \int_1^4\left[x\left[\ln(y)\right]_1^2 + \frac{1}{x}\left[\frac{y^2}{2}\right]_1^2\right]dx$$

$$= \int_1^4\left[x\left[\ln(2)-\ln(1)\right] + \frac{1}{2x}(4-1)\right]dx$$

$$= \int_1^4\left[x\ln(2) + \frac{3}{2x}\right]dx = \int_1^4 \ln(2)\,x\,dx + \int_1^4 \frac{3}{2}\frac{1}{x}\,dx$$

$$= \ln(2)\left[\frac{x^2}{2}\right]_1^4 + \frac{3}{2}\left[\ln(x)\right]_1^4$$

$$= \ln(2)\left[8-\frac{1}{2}\right] + \frac{3}{2}\left[\ln(4)-\ln(1)\right]$$

$$= \ln(2)\,\frac{15}{2} + \frac{3}{2}\ln(4) = \ln(2)\,\frac{15}{2} + \ln(2)\,\frac{6}{2} = \ln(2)\left[\frac{15}{2}+\frac{6}{2}\right] = \boxed{\ln(2)\,\frac{21}{2}}$$

(13) $$\int_0^2\int_0^\pi r\sin^2\theta\,d\theta\,dr = \int_0^2 r\left[\int_0^\pi \sin^2\theta\,d\theta\right]dr = \int_0^2 r\left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_0^\pi dr$$

$$= \int_0^2 r\left[\left(\frac{\pi}{2} - \cancel{\frac{\sin(2\pi)}{4}}^{\,0}\right) - \left(\cancel{\frac{0}{2} - \frac{\sin(0)}{4}}^{\,0}\right)\right]dr = \int_0^2 r\left[\frac{\pi}{2}\right]dr = \frac{\pi}{2}\int_0^2 r\,dr$$

$$= \frac{\pi}{2}\left[\frac{r^2}{2}\right]_0^2 = \frac{\pi}{4}(4-0) = \boxed{\pi}$$

(27) Find the volume of the solid lying under the elliptic paraboloid $\frac{x^2}{4} + y^2/9 + z = 1$ and above the rectangle $R = [-1,1] \times [-2,2]$

R

x from -1 to 1
y from -2 to 2

Since $\frac{x^2}{4}+\frac{y^2}{9}+z=1$, define

$$z = f(x,y) = 1 - \frac{x^2}{4} - \frac{y^2}{9}$$

the desired area is:

$$A = \int_{-2}^{2}\int_{-1}^{1} 1-\frac{x^2}{4}-\frac{y^2}{9}\,dx\,dy = \int_{-2}^{2}\left[\left(x-\frac{x^3}{12}-\frac{xy^2}{9}\right)\right]_{-1}^{1} dy$$

$$= \int_{-2}^{2}\left(1-\frac{1}{12}-\frac{y^2}{9}\right)-\left(-1+\frac{1}{12}+\frac{y^2}{9}\right)dy$$

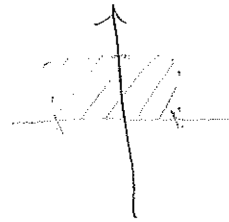
$$= \int_{-2}^{2} 1+1-\frac{1}{12}-\frac{1}{12}-\frac{y^2}{9}-\frac{y^2}{9}\,dy = \int_{-2}^{2} 2-\frac{2}{12}-\frac{2y^2}{9}\,dy$$

$$= \left[2y-\frac{2}{12}y-\frac{2}{27}y^3\right]_{-2}^{2} = \left(4-\frac{4}{12}-\frac{2^4}{27}\right)-\left(-4+\frac{4}{12}+\frac{2^4}{27}\right)$$

$$= 4+4-\frac{4}{12}-\frac{4}{12}-\frac{2^4}{27}-\frac{2^4}{27} = 8-\frac{8}{12}-\frac{2^5}{27} = 8-\frac{2}{3}-\frac{2^5}{27} = \frac{22}{3}-\frac{2^5}{27}$$

$$= \frac{9\times22-2^5}{27} = \frac{198-32}{27} = \boxed{\frac{166}{27}}$$

(37) Use symmetry to evaluate the double integral



$$\iint_R \frac{xy}{1+x^4}\,dA, \quad R=\{(x,y) \mid -1 \le x \le 1,\ 0 \le y \le 1\}$$

Is the function ~~odd~~ even with respect to x?

even : $f(-x,y) = \dfrac{-xy}{1+x} \neq f(x,y) = \dfrac{xy}{1+x^4}$, is not even

odd : $f(-x,y) = \dfrac{-xy}{1+x} = -f(x,y) \Rightarrow f(x,y)$ is odd with respect to x.

Hence,

$$\iint_R \frac{xy}{1+x^4} = \int_0^1\int_{-1}^1 \frac{xy}{1+x^4}\,dx\,dy \quad \text{But} \quad \int_{-1}^1 \frac{xy}{1+x^4}\,dx = 0$$

Hence $\boxed{\iint_R \frac{xy}{1+x^4} = 0}$

(Similar to 37-38) Use symmetry to evaluate the double integral.

$$\iint_R 2\sin(x) - 3y^3 + 5\,dA \qquad R=\{(x,y)\in\mathbb{R}^2 : -1 \le x \le 1, -1 \le y \le 1\}$$

Using linearity:

$$\iint_R 2\sin(x) - 3y^3 + 5\,dA = \underbrace{\overbrace{\iint_R 2\sin(x)\,dA}^{f_1}}_{V_1} - \underbrace{\overbrace{\iint_R 3y^3\,dA}^{f_2}}_{V_2} + \underbrace{\overbrace{\iint_R 5\,dA}^{f_3}}_{V_3}$$

f_1 is an odd function w.r.t. x since $f_1(-x) = 2\sin(-x) = -2\sin(x) = f(x)$.
the domain of integration is symmetric about the origin w.r.t x $\Rightarrow V_1 = 0$

f_2 is an odd function w.r.t. y since $f_2(-y) = 3(-y)^3 = -3y^3 = -f_2(y)$
the domain of integration is symmetric about the origin wrt y $\Rightarrow V_2 = 0$

So the volume is:

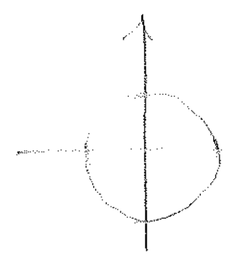
$$\int_{-1}^1\int_{-1}^1 5\,dx\,dy = 5(1-(-1))(1-(-1)) = 5\cdot 2\cdot 2 = \boxed{20}$$

- If f is odd in the variable x, and the domain of integration is symmetric about the y-axis, then the integral is zero

- If f is odd in the variable y, and the domain of integration is symmetric about the x-axis, then the integral is zero

$$\iint_R x^3y^2 + \ln(x^2+x+1)\sin(y^3)\,dA \;;\; R = \{(x,y) : x^2+y^2 \le 1\}$$

$$= \overbrace{\iint_R \underbrace{x^3y^2}_{f_1}\,dA}^{V_1} + \overbrace{\iint_R \underbrace{\ln(x^2+x+1)\sin(y^3)}_{f_2}\,dA}^{V_2}$$



R is symmetric about both x and y.

Since $f_1(-x,y) = (-x^3)y^2 = -x^3y^2 = -f_1(x,y)$, f_1 is an odd function

Hence $V_1 = 0$

Also,

since $f_2(x,-y) = \ln(x^2+x+1)\sin((-y)^3) = \ln(x^2+x+1)\sin(-y^3)$

since $\boxed{\sin(-x) = -\sin(x)}$ $= -\ln(x^2+x+1)\sin(y^3) = -f_2(x,y)$

f_2 is an odd function

Hence $V_2 = 0$

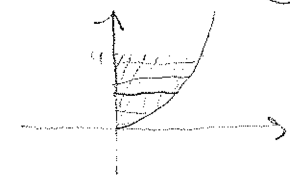
$$\Rightarrow \iint_R x^3y^2 + \ln(x^2+x+1)\sin(y^3)\,dA = V_1 + V_2 = 0 + 0 = \boxed{0}$$

$f(x) = y$ is s.t. $f(-x) = -f(x)$

$V - V = 0$

SECTION 15.3 :

$\sqrt{y} = x$
$\Rightarrow y = x^2$



(1) Evaluate the iterated integral:

$$\int_0^4\int_0^{\sqrt{y}} xy^2\,dx\,dy = \int_0^4 y^2\int_0^{\sqrt{y}} x\,dx\,dy = \int_0^4 y^2\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy = \int_0^4 y^2\,\frac{y}{2}\,dy$$

$$\frac{1}{2}\int_0^4 y^3\,dy = \frac{1}{2}\left[\frac{y^4}{4}\right]_0^4 = \frac{1}{2}\,\frac{4^4}{4} = \frac{1}{2}\,4^3 = \frac{64}{2} = \boxed{32}$$

Similar to (1–6)

$$\int_0^2\int_{x^2}^{2x}(8x+10y)\,dy\,dx = \int_0^2\left[\int_{x^2}^{2x}8x\,dy + \int_{x^2}^{2x}10y\,dy\right]dx$$

$$= \int_0^2\left[8x\int_{x^2}^{2x}dy + 10\int_{x^2}^{2x}y\,dy\right]dx = \int_0^2\left[8x[y]_{x^2}^{2x} + 10\left[\frac{y^2}{2}\right]_{x^2}^{2x}\right]dx$$

$$= \int_0^2 8x(2x-x^2) + 10\left(\frac{4x^2}{2} - \frac{x^4}{2}\right)dx$$

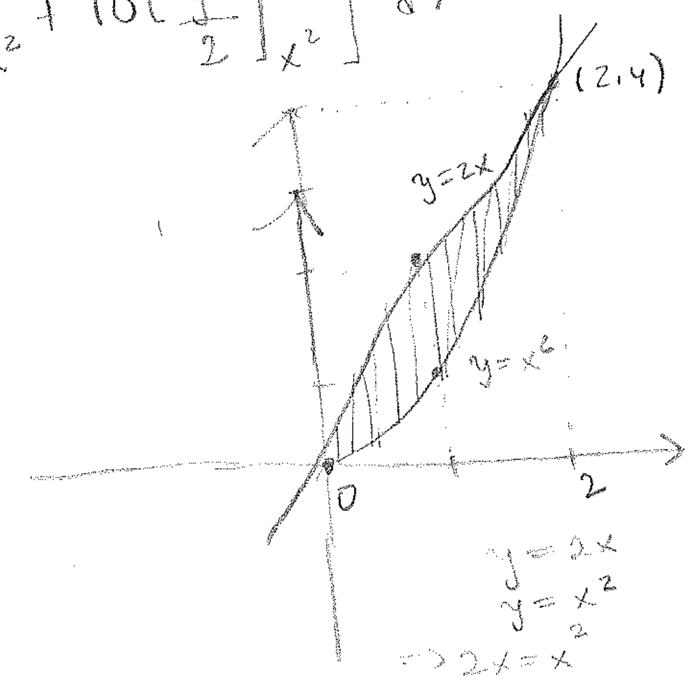
$$= \int_0^2 16x^2 - 8x^3 + \frac{40x^2 - 10x^4}{2}\,dx$$

$$= \left[16\frac{x^3}{3} - 8\frac{x^4}{4} + \frac{1}{2}\left(\frac{40x^3}{3} - \frac{10x^5}{5}\right)\right]_0^2$$

$$= \frac{128}{3} - \frac{128}{4} + \frac{1}{2}\left(\frac{320}{3} - \frac{320}{5}\right)$$

$$= \frac{4\times128 - 3\times128}{12} + \frac{1}{2}\left(\frac{5\times320 - 3\times320}{15}\right)$$

$$= \frac{128}{12} + \frac{640}{30} = \frac{2^7}{2^2\cdot3} + \frac{640}{5\cdot3\cdot2} = \frac{640+1280}{60} = \frac{1920}{60} = \boxed{32}$$



$y = 2x$
$y = x^2$
$\Rightarrow 2x = x^2$

$x = 2$ or $x = 0$

⑦ Evaluate the double integral

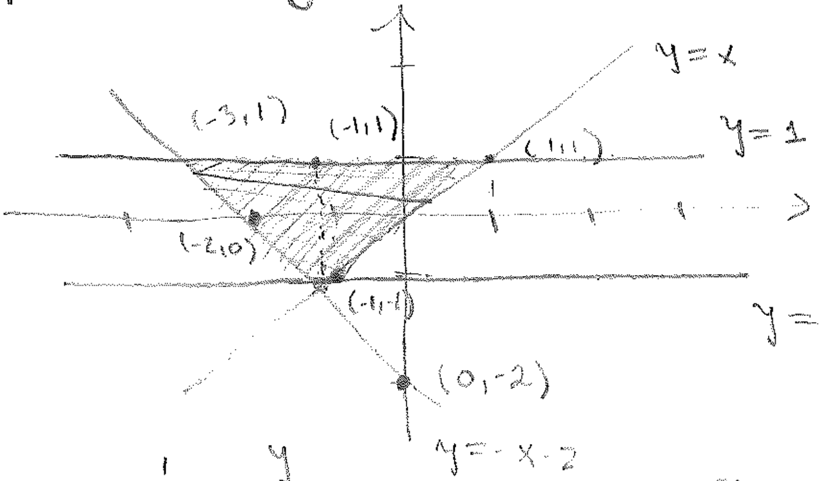
$$\iint_D y^2\,dA, \quad D=\{(x,y)\mid -1\le y\le 1,\ -y-2\le x\le y\}$$

First, graph the region D.

$$-y-2\le x\le y \Rightarrow \begin{cases} x=y \\ x=-y-2 \Rightarrow y=-x-2 \end{cases}$$

If $y=1$
$1=-x-2$
$3=-x$
$\Rightarrow x=-3$

$$-1\le y\le 1 \Rightarrow \begin{cases} y=1 \\ y=-1 \end{cases}$$



$$\int_{-1}^{1}\int_{-y-2}^{y} y^2\,dx\,dy = \int_{-1}^{1} y^2\int_{-y-2}^{y} dx\,dy = \int_{-1}^{1} y^2\,[x]_{-y-2}^{y}\,dy = \int_{-1}^{1} y^2(y-(-y-2))\,dy$$

$$= \int_{-1}^{1} y^2(y+y+2)\,dy = \int_{-1}^{1} y^2(2y+2)\,dy = \int_{-1}^{1} 2y^3+2y^2\,dy$$

$$= \left[2\frac{y^4}{4}+2\frac{y^3}{3}\right]_{-1}^{1} = \left(\frac12+\frac23\right)-\left(\frac12-\frac23\right) = \boxed{\frac43}$$

Alternatively: changing the order of integration

$$\int_{-3}^{-1}\int_{-x-2}^{1} y^2\,dy\,dx + \int_{-1}^{1}\int_{x}^{1} y^2\,dy\,dx = \int_{-3}^{-1}\left[\frac{y^3}{3}\right]_{-x-2}^{1} dx + \int_{-1}^{1}\left[\frac{y^3}{3}\right]_{x}^{1} dx$$

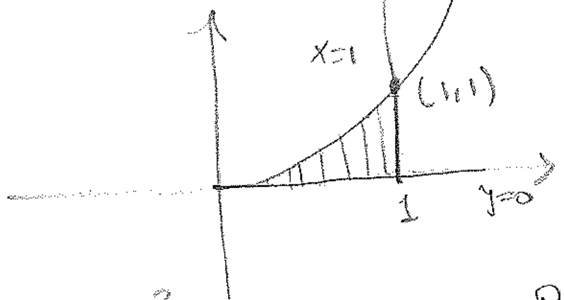
$$= \int_{-3}^{-1} \frac{(-x-2)^3}{3}-\frac13\,dx + \int_{-1}^{1} \frac13-\frac{x^3}{3}\,dx = \int_{-3}^{-1} \frac{-x^3-3x^2(2)-3x(-2)^2+(2)^3-1}{3}\,dx + \int_{-1}^{1} \frac{1-x^3}{3}\,dx$$

$$= \int_{-3}^{-1} -x^3+6x^2 \quad \cdots$$

17 Evaluate the double integral

$$\iint_D x\cos y \, dA \quad , \quad D \text{ is bounded by } y=0,\ y=x^2,\ x=1$$

First, graph the region:



$$\int_0^1\int_{\sqrt{y}}^1 x\cos y \, dx\,dy = \int_0^1\int_0^{x^2} x\cos y \, dy\,dx$$

Pick the easiest order of integration

$$\int_0^1\int_0^{x^2} x\cos y \, dy\,dx = \int_0^1 x \int_0^{x^2} \cos y \, dy\,dx = \int_0^1 x\,[\sin y]_0^{x^2} dx$$

$$= \int_0^1 x(\sin(x^2) - \sin(0))\,dx = \int_0^1 x\sin(x^2)\,dx$$ requires integration by substitution

$$x^2 = u \Rightarrow 2x\,dx = du \Rightarrow \int_0^1 x\sin(x^2)\,dx = \int_0^{\square} \frac{du}{2}\sin(u)$$

$$= \frac{1}{2}\int_{\square}^{\square} \sin(u)\,du = \frac{1}{2}[-\cos(u)]_{\square}^{\square} \rightsquigarrow \frac{1}{2}[-\cos(x^2)]_0^1 = \frac{1}{2}[-\cos(1) - (-\cos(0))]$$

$$= \frac{1}{2}[1-\cos(1)]$$. Using the other order:

$$\int_0^1\int_{\sqrt{y}}^1 x\cos y \, dx\,dy = \int_0^1 \cos y \int_{\sqrt{y}}^1 x\,dx\,dy = \int_0^1 \cos y \left[\frac{x^2}{2}\right]_{\sqrt{y}}^1 dy = \int_0^1 \cos y\left(\frac{1}{2} - \frac{y}{2}\right)dy$$

$$= \int_0^1 \frac{\cos y}{2} - \frac{\cos y(y)}{2}\,dy = \frac{1}{2}\int_0^1 \cos y - y\cdot\cos y\,dy = \frac{1}{2}\Big[\overbrace{\int_0^1 \cos y\,dy}^{A_1} - \overbrace{\int_0^1 y\cos y\,dy}^{A_2}\Big]$$

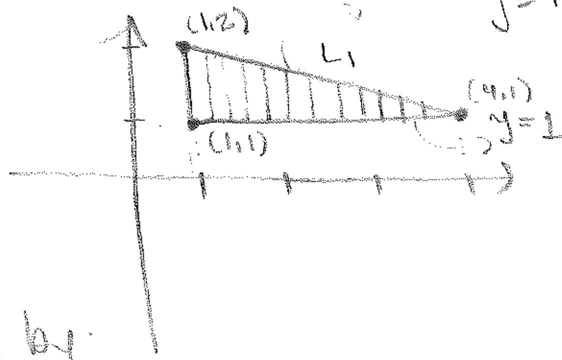
$$A_1 = [\sin y]_0^1 = \sin(1) - \sin(0) = \sin(1)$$

A_2 by parts ... (more difficult!)

(25) Find the volume of the given solid.

Under the surface $z = xy$ and above the triangle with vertices $(1,1)$, $(4,1)$ and $(1,2)$.

First, graph the domain



This line is given by $y = mx + b$:

$2 = m(1) + b$

$1 = m(4) + b$

$-7 = -3b \Rightarrow b = 7/3$

$\Rightarrow 2 = m + 7/3$

$\Rightarrow m = 2 - \frac{7}{3} = -\frac{1}{3}$

$\Rightarrow L_1: y = -\frac{1}{3}x + 7/3$

the Volume is given by:

$$\int_1^4 \int_1^{-\frac{1}{3}x + 7/3} xy \, dy \, dx = \int_1^4 x \left[\int_1^{-\frac{1}{3}x+7/3} y \, dy \right] dx$$

$$= \int_1^4 x \left[\frac{y^2}{2} \right]_1^{-\frac{1}{3}x + 7/3} dx = \frac{1}{2} \int_1^4 x\left[\left(-\frac{1}{3}x + 7/3\right)^2 - 1\right] dx$$

$$= \frac{1}{2} \int_1^4 x\left[\frac{x^2}{9} - \frac{14x}{9} + \frac{49}{9} - 1\right] dx = \frac{1}{2}\int_1^4 x\left[\frac{x^2}{9} - \frac{14x}{9} + \frac{40}{9}\right] dx$$

$$= \frac{1}{18}\int_1^4 x^3 - 14x^2 + 40x \, dx = \frac{1}{18}\left[\frac{x^4}{4} - \frac{14x^3}{3} + 20x^2\right]_1^4$$

$$= \frac{1}{18}\left[\left(\frac{4^4}{4} - \frac{14(4)^3}{3} + 20(4)^2\right) - \left(\frac{1}{4} - \frac{14}{3} + 20\right)\right]$$

$$= \frac{1}{18}\left[4^3 - \frac{1}{4} - 4^3\frac{14}{3} + \frac{1 \cdot 14}{3} + 16(20) - 1(20)\right]$$

$$= \frac{1}{18}\left[\frac{4^4 - 1}{4} + \frac{14}{3}(1 - 4^3) + 15(20)\right]$$

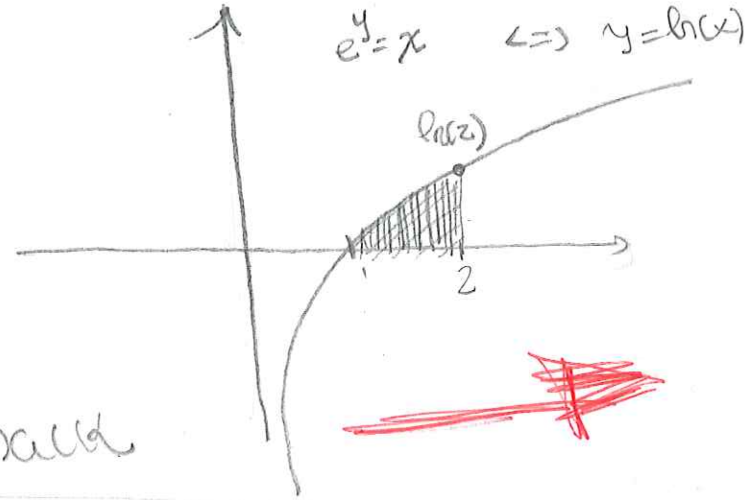
$$= \frac{1}{18}\left[\frac{255}{4} - 294 + 300\right] = \frac{1}{18}\left[\frac{255}{4} + 6\right] = \frac{1}{18}\left[\frac{279}{4}\right] = \frac{279}{72}$$

$$\frac{93}{24} = \boxed{\frac{31}{8}}$$

(47) Sketch the region of integration and change the order of integration

$$\int_1^2 \int_0^{\ln(x)} f(x,y)\,dy\,dx$$

$$= \int_0^{\ln(2)} \int_{e^y}^{2} f(x,y)\,dx\,dy$$

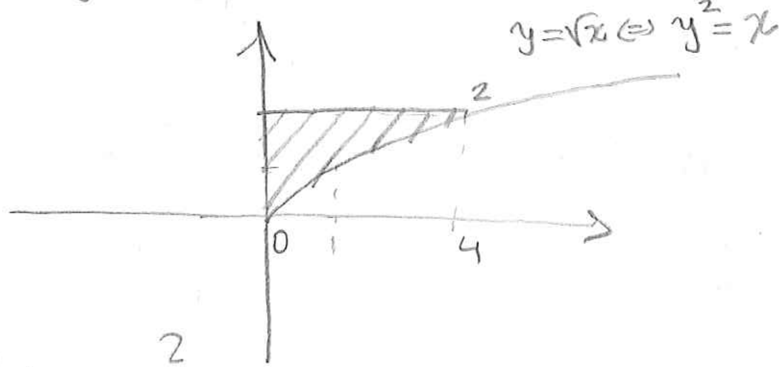


Do the one on the back

(51) Evaluate the integral by reversing the order of integration:

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx =$$

$$\int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy$$



$$= \int_0^2 \frac{1}{y^3+1} \int_0^{y^2} dx\,dy = \int_0^2 \frac{1}{y^3+1} \left[x\right]_0^{y^2} dy = \int_0^2 \frac{y^2}{y^3+1}\,dy$$

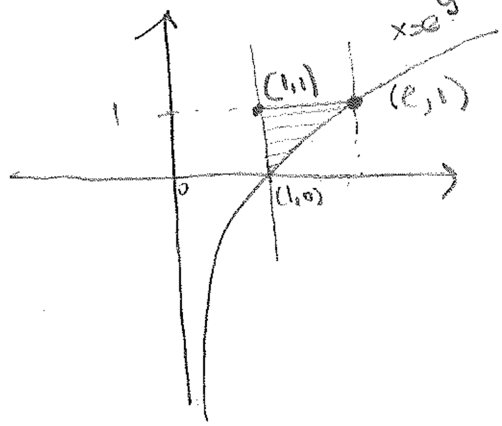
Substitution: $y^3+1 = u \Rightarrow 3y^2 = du \Rightarrow y^2 = \frac{du}{3}$

$$\int_0^2 \frac{y^2}{y^3+1}\,dy = \int_0^{\square} \frac{\frac{du}{3}}{u} = \int_0^{\square} \frac{du}{3u} = \frac{1}{3}\int_0^{\square} \frac{du}{u} = \frac{1}{3}\left[\ln(u)\right]_0^{\square}$$

$$\rightsquigarrow \frac{1}{3}\left[\ln(y^3+1)\right]_0^2 = \frac{1}{3}\left[\ln(9) - \ln(1)\right] = \frac{\ln(9)}{3} = \boxed{\frac{1}{3}\ln(9)}$$

Change the order of integration in the following integral

$$\int_0^1 \int_1^{e^y} f(x,y)\,dx\,dy$$



$x_h = e^y \Leftrightarrow \ln(x_h) = y$

$x_\ell = 1$

$y = \ln(x) = 1$

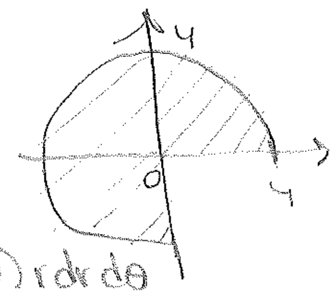
$\Rightarrow x = e$

$$\int_1^e \int_{\ln(x)}^1 f(x,y)\,dy\,dx$$

Similar to 43-48

SECTION 15.4

(1) Use polar coordinates!

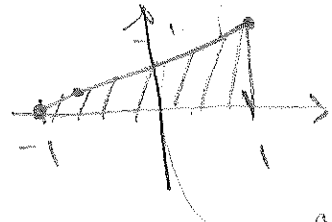


$$\iint_R f(x,y)\,dA \qquad \int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$1 = m + \frac{1}{2}$

$\Rightarrow m = 1/2$

(3) Use rectangular coordinates!



$y = mx + b$

$1 = m + b$

$0 = -m + b$

$1 = 0 + 2b \Rightarrow b = \frac{1}{2}$

$\Rightarrow y = \frac{x}{2} + \frac{1}{2} = \frac{x+1}{2}$

$\Rightarrow 2y - 1 = x$

$$\iint_R f(x,y)\,dA = \int_{-1}^{1} \int_0^{\frac{x+1}{2}} f(x,y)\,dy\,dx$$

$$= \int_0^1 \int_{2y-1}^{1} f(x,y)\,dx\,dy$$

(9) Evaluate the given integral by changing to polar coordinates.

$$\iint_R \sin(x^2+y^2)\,dA, \qquad R =$$

$$\iint_R f(x,y)\,dA = \int_0^{\pi/2} \int_1^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta = \int_0^{\pi/2} \int_1^3 \sin((r\cos\theta)^2 + (r\sin\theta)^2)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_1^3 \sin(r^2)\,r\,dr\,d\theta$$

Change vars $u = r^2 \quad du = 2r\,dr \Rightarrow \frac{du}{2} = r\,dr \Rightarrow \int_0^{\pi/2} \int_1^9 \frac{\sin(u)\,du}{2}\,d\theta$

$$= \int_0^{\pi/2} \left[-\frac{\cos(r^2)}{2} \right]_1^3 d\theta = \frac{1}{2} \int_0^{\pi/2} -\cos(9) + \cos(1)\,d\theta = \frac{1}{2}(-\cos 9 + \cos 1)\left(\theta\right]_0^{\pi/2} = \frac{\pi}{4}(\cos(1) - \cos(9))$$

SECTION 15.7:

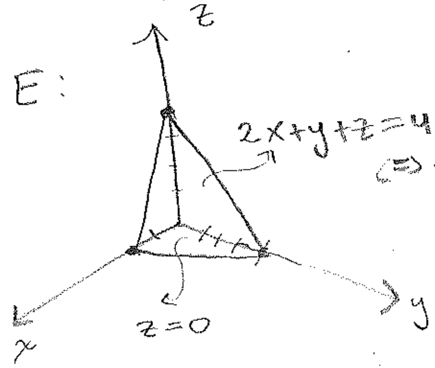
19 Use a triple integral to find the volume of the given solid.
The tetrahedron enclosed by the coordinate planes and the plane

$$2x+y+z=4$$

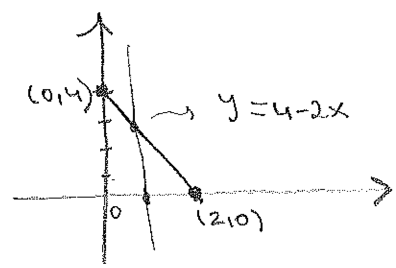
First, graph the solid and region

If $z=0$
then $2x+y+z=4$
$\Leftrightarrow 2x+y=4$
$\Leftrightarrow y=4-2x$

E:



D:



Second, set up the integral

$$V(E) = \iiint_E dV = \int_0^2\int_0^{4-2x}\int_0^{4-2x-y} dz\,dy\,dx$$

third, Evaluate:

$$= \int_0^2\int_0^{4-2x}\left[z\right]_0^{4-2x-y} dy\,dx = \int_0^2\int_0^{4-2x} 4-2x-y\ dy\,dx$$

$$= \int_0^2 \left(4y-2xy-\frac{y^2}{2}\right)_0^{4-2x} dx = \int_0^2 4(4-2x)-2x(4-2x)-\frac{(4-2x)^2}{2}\,dx$$

$$= \int_0^2 16-8x-8x+4x^2-\left(\frac{16-16x+4x^2}{2}\right)dx$$

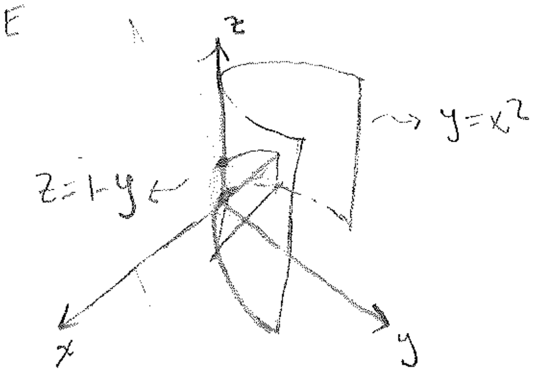
$$= \int_0^2 \cancel{16}-\cancel{16x}+\cancel{4x^2}-8+\cancel{8x}-\cancel{2x^2}\,dx = \int_0^2 2x^2-8x+8\,dx = 2\int_0^2 x^2-4x+4\,dx$$

$$= 2\left(\frac{x^3}{3}-2x^2+4x\right)_0^2 = 2\left(\frac{8}{3}-\cancel{8}+\cancel{8}\right) = \boxed{\frac{16}{3}}$$

21) the solid enclosed by the cylinder $y=x^2$ and the planes $z=0$ and $y+z=1$.

First, graph the solid and region

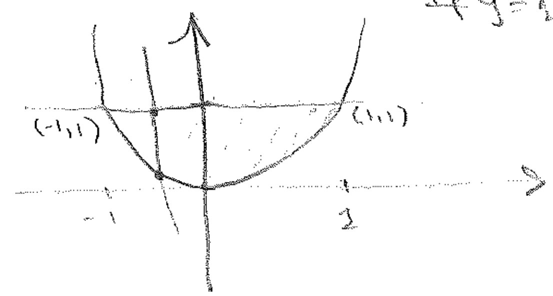
E



Region D

If $z=0$ then $y=1$

If $y=1 \Rightarrow x=\pm 1$



Second, set up the triple integral

$$V(E)=\iiint_E dV=\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx$$

third, evaluate the integral:

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx=\int_{-1}^{1}\int_{x^2}^{1}\left[z\right]_0^{1-y} dy\,dx=\int_{-1}^{1}\int_{x^2}^{1} 1-y\,dy\,dx$$

$$=\int_{-1}^{1}\left(y-\frac{y^2}{2}\right)_{x^2}^{1} dx=\int_{-1}^{1}\left(1-\frac{1}{2}\right)-\left(x^2-\frac{x^4}{2}\right)dx=\int_{-1}^{1}\frac{1}{2}-x^2+\frac{x^4}{2}\,dx$$

$$=\left(\frac{1}{2}x-\frac{x^3}{3}+\frac{x^5}{10}\right)_{-1}^{1}=\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{10}\right)-\left(-\frac{1}{2}+\frac{1}{3}-\frac{1}{10}\right)$$

$$=\frac{1}{2}+\frac{1}{2}-\frac{1}{3}-\frac{1}{3}+\frac{1}{10}+\frac{1}{10}$$

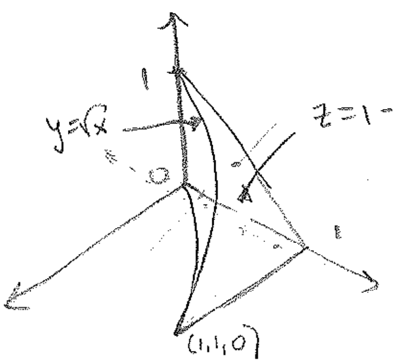
$$=1-\frac{2}{3}+\frac{1}{5}=\frac{15-10+3}{15}=\boxed{\frac{8}{15}}$$

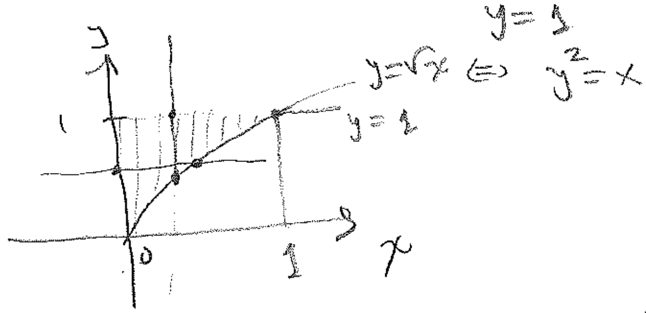
(33) $\displaystyle\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$

Rewrite this integral as an equivalent iterated integral in the five other orders: $dz\,dx\,dy$, $dy\,dz\,dx$, $dy\,dx\,dz$, $dx\,dy\,dz$, $dx\,dz\,dy$

First, graph the solid and Domain:

If $z=0$ then $y=1$





$z=1-y$; $y=\sqrt{x}$

$(z=1-\sqrt{x})$

SECOND, set up the integral

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$$

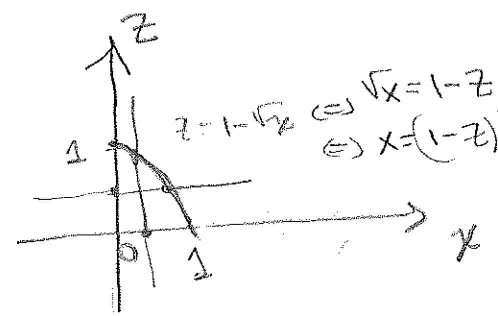
$$=\int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy$$

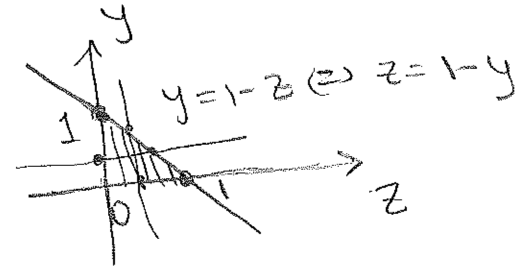
$$=\int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$$

$$=\int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$$

$$=\int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$$

$$=\int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy$$

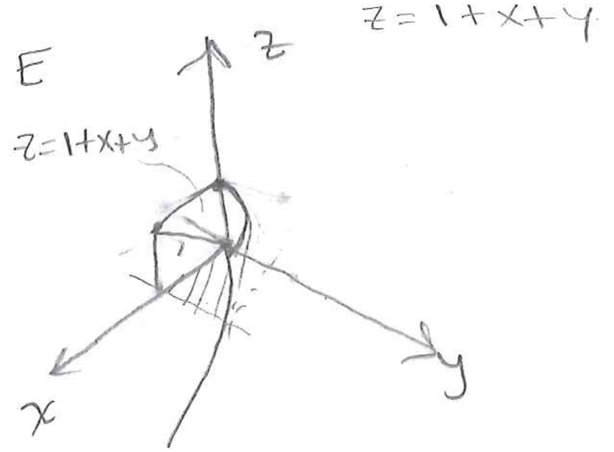


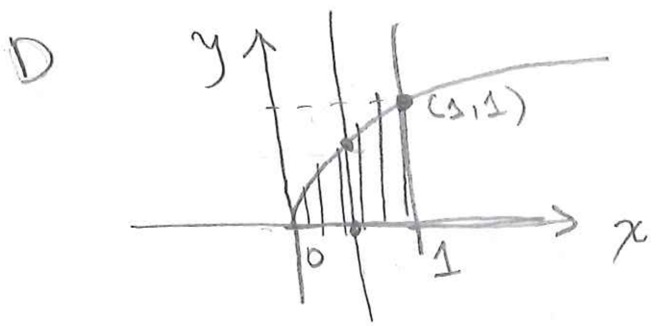


13 Evaluate the triple integral   First Exercise

$\iiint_E 6xy\, dV$, where E lies under the plane $z = 1+x+y$ and above the region in the xy-plane bounded by $y=\sqrt{x}$, $y=0$, $x=1$

First, graph the solid and region.





SECOND, SET UP THE INTEGRAL:

$$\iiint_E 6xy\,dV = \int_0^1\int_0^{\sqrt{x}}\int_0^{1+x+y} 6xy\,dz\,dy\,dx$$

third, Evaluate the integral:

$$\int_0^1\int_0^{\sqrt{x}}\int_0^{1+x+y} 6xy\,dz\,dy\,dx = \int_0^1\int_0^{\sqrt{x}} 6xy(1+x+y)\,dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{x}} 6xy+6x^2y+6xy^2\,dy\,dx = \int_0^1 \left(3xy^2+3x^2y^2+2xy^3\right)\Big|_0^{\sqrt{x}}\,dx$$

$$= \int_0^1 3x^2+3x^3+2(x)^{5/2}\,dx = \left(x^3+\frac{3}{4}x^4+\frac{4}{7}(x)^{7/2}\right)\Big|_0^1$$

$$= 1+\frac{3}{4}+\frac{4}{7} = \frac{28+21+16}{28} = \boxed{\frac{65}{28}}$$

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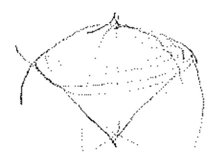
25 Use polar coordinates to find the volume of the given solid. Above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$.

First, find the domain of integration

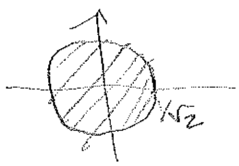
For that we need the intersection:

$$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=1 \iff 2x^2+2y^2=1 \iff x^2+y^2=\frac{1}{2}$$

circle of radius $\frac{1}{\sqrt{2}}$



ice-cream cone region



Second, set up the integral. We need to substract the volume of the sphere from the volume

$$\iint_R \sqrt{1-x^2-y^2}-\sqrt{x^2+y^2}\,dA \xrightarrow[\text{polar}]{\text{change to}} \int_0^{2\pi}\int_0^{1/\sqrt{2}} f(r\cos\theta, y\cos\theta)\, r\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{1/\sqrt{2}}\left(\sqrt{1-r^2}-r\right)r\,dr\,d\theta=\int_0^{2\pi}\int_0^{1/\sqrt{2}} \underbrace{r\sqrt{1-r^2}}_{A}-\underbrace{r^2}_{B}\,dr\,d\theta$$

$$A=\int_0^{1/\sqrt{2}} r\sqrt{1-r^2}\,dr, \text{ sub. } u=1-r^2 \quad du=-2r\,dr \Rightarrow -\frac{du}{2}=r\,dr$$

$$=\int_0^{1/2}\frac{u^{1/2}}{-2}\,du=\frac{2}{-2\cdot 3}\left[u^{3/2}\right]_0^{1}=-\frac{1}{3}\left[(1-r^2)^{3/2}\right]_0^{1/\sqrt{2}}=-\frac{1}{3}\left(\left(\frac{1}{2}\right)^{3/2}-1\right)$$

$$=\frac{1}{3}-\frac{1}{3}\left(\frac{1}{2}\right)^{3/2}$$

$$B=\int_0^{1/\sqrt{2}} r^2\,dr=\left[\frac{r^3}{3}\right]_0^{1/\sqrt{2}}=\frac{\left(\frac{1}{\sqrt{2}}\right)^3}{3}=\frac{\left(\frac{1}{2}\right)^{3/2}}{3}=\frac{1}{3}\cdot\left(\frac{1}{2}\right)^{3/2}$$

$$\rightsquigarrow \int_0^{2\pi} A-B\,d\theta=\int_0^{2\pi}\frac{1}{3}-\frac{1}{3}\left(\frac{1}{2}\right)^{3/2}-\frac{1}{3}\left(\frac{1}{2}\right)^{3/2}d\theta=\frac{1}{3}\left(1-\frac{2}{2^{3/2}}\right)\int_0^{2\pi}d\theta$$

$$=\frac{1}{3}\left(1-\frac{1}{\sqrt{2}}\right)(2\pi)=\boxed{\frac{2}{3}\pi\left(1-\frac{1}{\sqrt{2}}\right)}$$

(31) Evaluate the iterated integral by converting to polar coordinates.

$$\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\,dx\,dy$$

$x = y$

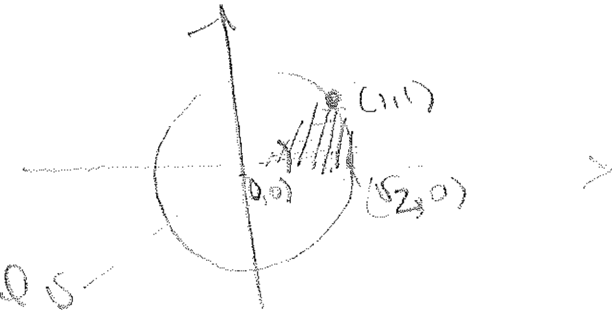
$x = \sqrt{2-y^2}$

$x^2 = 2-y^2$

$x^2+y^2 = 2$

$\Rightarrow$ circle centered at the origin with radius $\sqrt{2}$

the region is :



So the polar integral is

$$\int_0^{\pi/4}\int_0^{\sqrt{2}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^{\sqrt{2}} (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta = \int_0^{\pi/4}\int_0^{\sqrt{2}} r^2\cos\theta + r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^{\sqrt{2}} r^2(\cos\theta + \sin\theta)\,dr\,d\theta = \int_0^{\pi/4} \cos\theta + \sin\theta \int_0^{\sqrt{2}} r^2\,dr\,d\theta$$

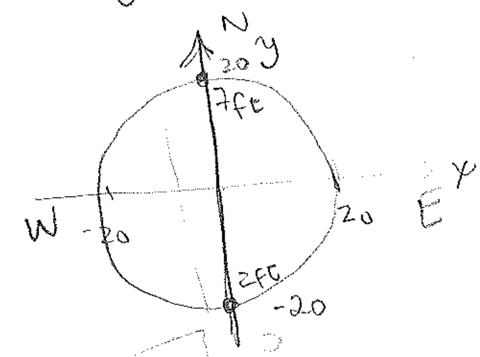
$$= \int_0^{\pi/4} \cos\theta + \sin\theta \left[\frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta = \int_0^{\pi/4} \cos\theta + \sin\theta \left(\frac{2\sqrt{2}}{3}\right) d\theta$$

$$= \frac{2\sqrt{2}}{3}\left(\int_0^{\pi/4} \cos\theta\,d\theta + \int_0^{\pi/4} \sin\theta\,d\theta\right) = \frac{2\sqrt{2}}{3}\left(\left[\sin\theta\right]_0^{\pi/4} + \left[-\cos\theta\right]_0^{\pi/4}\right)$$

$$= \frac{2\sqrt{2}}{3}\left(\sin\left(\frac{\pi}{4}\right) - \sin(0)^{0} - \cos\left(\pi/4\right) + \cos(0)\right)$$

$$= \frac{2\sqrt{2}}{3}\left(\cancel{\frac{\sqrt{2}}{2}} - \cancel{\frac{\sqrt{2}}{2}} + 1\right) = \boxed{\frac{2\sqrt{2}}{3}}$$

(35)



$f(x,y)$ = depth of the pool

the depth can be thought of as a plane. Two points on the plane are

$P=(0,-20,2)$, $Q(0,20,7)$

the depth of the pool increases linearly at a rate of $7ft - 2ft = 5ft \Rightarrow \frac{5ft}{40}$/units

So at the point $(20,0)$ we will have $2ft + \frac{5ft}{40 units} \cdot \frac{20 units}{1} = 2 + \frac{10}{4} = \frac{18}{4}$

So another point is $R=(20,0,\frac{18}{4})$.

Now we can find the plane:

$\overrightarrow{PQ} = (0,20,7) - (0,-20,2) = \langle 0,40,5 \rangle$

$\overrightarrow{PR} = (20,0,\frac{18}{4}) - (0,-20,2) = \langle 20,20,\frac{10}{4} \rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 40 & 5 \\ 20 & 20 & 10/4 \end{vmatrix} = \hat{i}(100-100) - \hat{j}(-100) + \hat{k}(-800)$$
$$= 100\hat{j} - 800\hat{k} = \vec{n}$$

TAKE AS the normal vector $\vec{n} = \hat{j} - 8\hat{k} = \langle 0,1,-8 \rangle$

the plane is: $0 = \vec{n} \cdot (\langle x,y,z \rangle - \langle 0,20,7 \rangle) = \langle 0,1,-8 \rangle \cdot \langle x, y-20, z-7 \rangle$

$\Rightarrow y - 20 - 8z + 56 = 0 \Rightarrow y - 8z + 36 = 0 \Rightarrow z = \frac{-36-y}{-8}$

$\boxed{z = \frac{y+36}{8}}$ Check that, if $y=20 \Rightarrow z=7$, if $y=-20 \Rightarrow z=2$

Now integrate: $\iint_R f(x,y)\,dA = \int_0^{2\pi}\int_0^{20} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

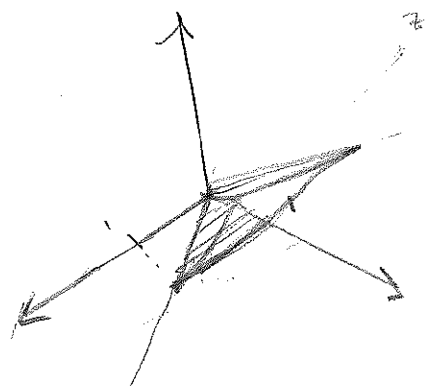
$$= \int_0^{2\pi}\int_0^{20} \frac{r^2\sin\theta + 36r}{8}\,dr\,d\theta = \frac{1}{8}\int_0^{2\pi} \frac{r^3}{3}\Big|_0^{20}\sin\theta + \frac{r^2}{2}\Big|_0^{20} 36\,d\theta = \frac{1}{8}\int_0^{2\pi} \frac{8000}{3}\sin\theta + 7200\,d\theta$$

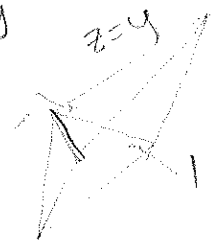
$$= \frac{1}{8}\left(\frac{8000}{3}\left[-\cos\theta\right]_0^{2\pi} + 7200\left[\theta\right]_0^{2\pi}\right) = \frac{1}{8}\left(\frac{8000}{3}(-1+1) + 7200(2\pi)\right) = \boxed{1800\pi}$$

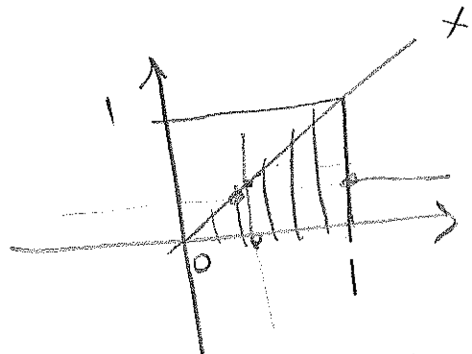
(35) Write five other iterated integrals that are equal to the given iterated integral:

$$\int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy \implies \begin{array}{l} z_{low}=0 \; ; \; z_{high}=y \\ x_{low}=y \; ; \; x_{high}=1 \\ y_{low}=0 \; ; \; y_{high}=1 \end{array}$$

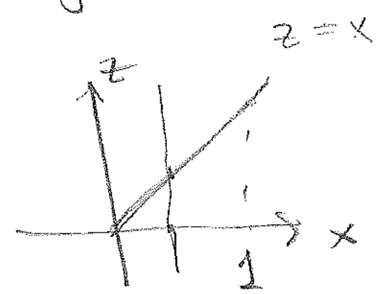
First, graph the solid and domain:







$dz\,dy\,dx$; $dz\,dx\,dy$
$dy\,dz\,dx$; $dy\,dx\,dz$
$dx\,dy\,dz$; $dx\,dz\,dy$



SECOND, SET UP THE INTEGRAL

$$\int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy$$

$$= \int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^x \int_z^x f(x,y,z)\,dy\,dz\,dx$$

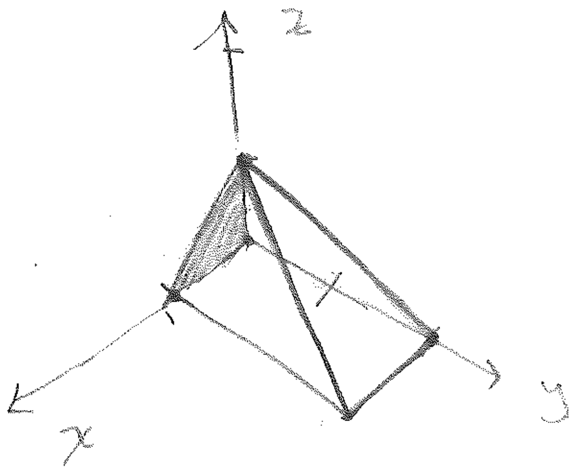
(27) Sketch the solid whose volume is given by the iterated integral.

$$\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\,dz\,dx$$

$y_{low} = 0$; $y_{high} = 2-2z$

$z_{low} = 0$; $z_{high} = 1-x$

$x_{low} = 0$; $x_{high} = 1$



For y:

$y = 2-2z$

$z=0 \Rightarrow y = 2$

$y=0 \Rightarrow 0 = 2-2z$

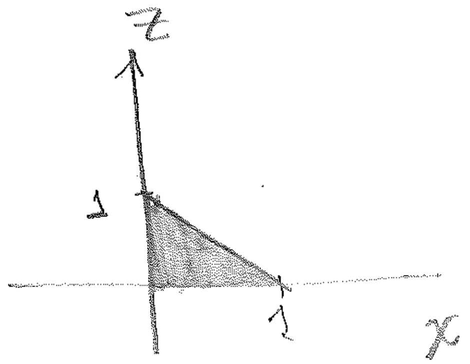
$2z = 2 \Rightarrow z = 1$

For z:

$z = 1-x$

$x=0 \Rightarrow z = 1$

$z=0 \Rightarrow 0 = 1-x \Rightarrow x = 1$



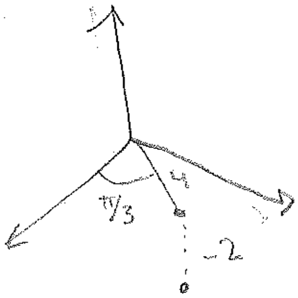
SECTION 15.8 : Cylindrical Coordinates

$\boxed{1}$ $x = r\cos\theta$; $y = r\sin\theta$; $z = z$

$\boxed{2}$ $r^2 = x^2 + y^2$; $\tan\theta = \frac{y}{x}$; $z = z$

(1) Plot the point whose cylindrical coordinates are given. then find the rectangular coordinates of the point

(a) $(4, \pi/3, -2)$



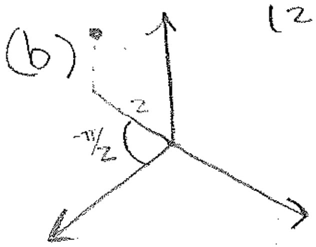
$x = r\cos(\theta) = 4\cdot\cos(\pi/3) = \frac{4}{2} = 2$

$y = r\sin(\theta) = 4\cdot\sin(\pi/3) = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$

$z = -2$

$\Rightarrow \boxed{(2, 2\sqrt{3}, -2)}$

(b) $(2, -\pi/2, 1)$



$x = r\cdot\cos(\theta) = 2\cdot\cos(-\pi/2) = 0$

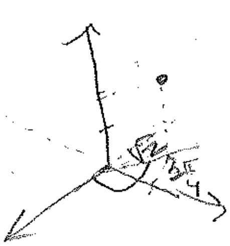
$y = r\cdot\sin(\theta) = 2\cdot\sin(-\pi/2) = 2\cdot(-1) = -2$

$z = 1$

$\Rightarrow \boxed{(0, -2, 1)}$

(2) Plot the point whose cylindrical coordinates are given. then find the rectangular coordinates of the point

(a) $(\sqrt{2}, 3\pi/4, 2)$

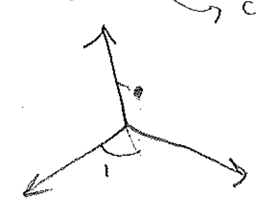


$x = r\cos\theta = \sqrt{2}\cdot\cos\left(\frac{3\pi}{4}\right) = \sqrt{2}\cdot\frac{-\sqrt{2}}{2} = \underline{-1}$

$y = r\sin\theta = \sqrt{2}\cdot\sin\left(\frac{3\pi}{4}\right) = \sqrt{2}\cdot\frac{\sqrt{2}}{2} = \underline{1}$

$z = 2 \Rightarrow \boxed{(-1, 1, 2)}$

(b) $(1, 1, 1)$ ↪ close to $\frac{\pi}{3}$ but less



$x = r\cos\theta = 1\cdot\cos(1) = \cos(1)$

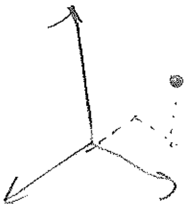
$y = r\sin\theta = 1\cdot\sin(1) = \sin(1)$

$z = 1$

$\Rightarrow \boxed{(\cos(1), \sin(1), 1)}$

(3) Change from rectangular to cylindrical coordinates.

(a) $(-1,1,1)$.



$x^2+y^2=r^2 \Rightarrow (-1)^2+(1)^2=r^2$

$\Rightarrow 2=r^2 \Rightarrow \boxed{r=\sqrt{2}}$

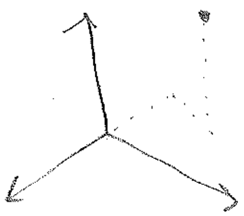
$\tan\theta=\dfrac{y}{x} \Rightarrow \theta=\arctan\left(\dfrac{1}{-1}\right)=\arctan(-1)$

$=-45$ degrees

$=\dfrac{3\pi}{4}$

$\Rightarrow \boxed{\left(\sqrt{2},\dfrac{3\pi}{4},1\right)}$

(b) $(-2,2\sqrt{3},3)$



$x^2+y^2=r^2 \Rightarrow (-2)^2+(2\sqrt{3})^2=r^2$

$\Rightarrow 4+12=r^2 \Rightarrow \boxed{r=4}$

$\tan\theta=\dfrac{y}{x} \Rightarrow \theta=\arctan\left(\dfrac{2\sqrt{3}}{-2}\right)=\arctan(-\sqrt{3})=\dfrac{2\pi}{3}$

$\Rightarrow \boxed{\left(4,\dfrac{2\pi}{3},3\right)}$

(6) Describe in words the surface whose equation is given

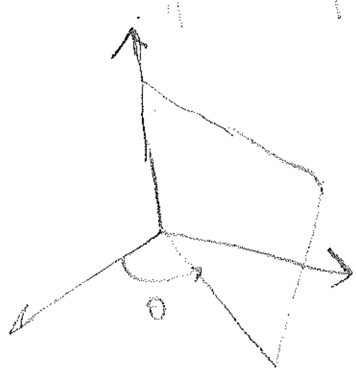
$r=$ Constant this is a cylinder of radius 5, centered at the origin

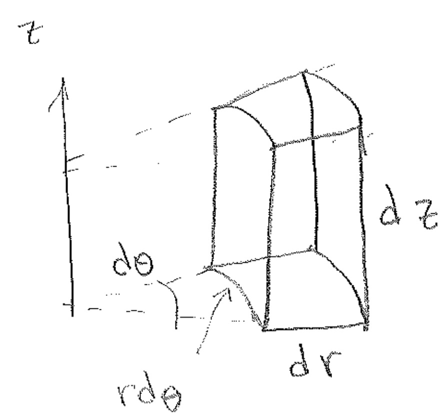
this is the key for cylindrical coordinates!

Cylindrical box

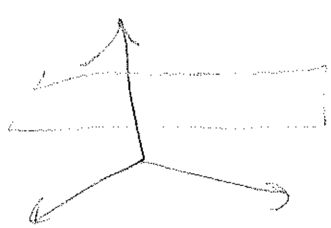
$r\,dz\,dr\,d\theta$

$\theta=$ Constant

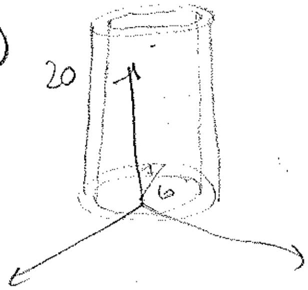




$z=$ constant



Area of base $r\,\Delta r\,\Delta\theta$

(13) 

$0 \leq z \leq 20$

$0 \leq \theta \leq 2\pi$

$6 \leq r \leq 7$

Cylindrical coordinates

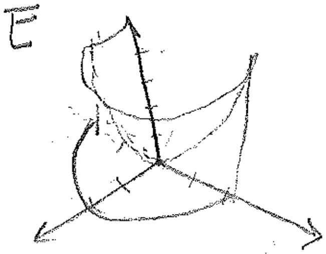
(15) Sketch the solid whose volume is given by the integral and evaluate the integral.

$$\int_{-\pi/2}^{\pi/2} \int_0^2 \int_0^{r^2} r \, dz \, dr \, d\theta$$

$z = 0$

$z = r^2$

$\Rightarrow z = y^2$

E 

$$= \int_{-\pi/2}^{\pi/2} \int_0^2 r [z]_0^{r^2} dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^2 r^3 \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_0^2 d\theta$$

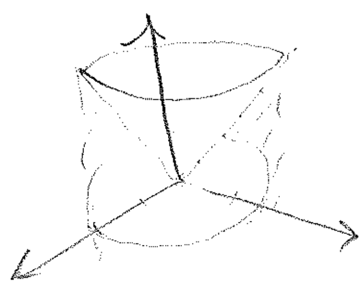
$$= \int_{-\pi/2}^{\pi/2} 4 \, d\theta = 4[\theta]_{-\pi/2}^{+\pi/2} = 4(\pi/2 - (-\pi/2)) = 4\pi$$

(16) $$\int_0^2 \int_0^{2\pi} \int_0^r r \, dz \, d\theta \, dr =$$

$z = 0$

$z = r$

$\to$ a cone



outside cone

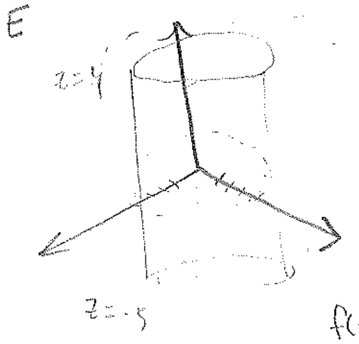
$$\int_0^2 \int_0^{2\pi} r [z]_0^r \, d\theta \, dr =$$

$$\int_0^2 \int_0^{2\pi} r^2 \, d\theta \, dr = \int_0^2 r^2 [\theta]_0^{2\pi} dr = \int_0^2 2\pi \cdot r^2 \, dr = 2\pi \int_0^2 r^2 \, dr$$

$$= 2\pi \left[\frac{r^3}{3}\right]_0^2 = 2\pi\left(\frac{8}{3}\right) = \boxed{\frac{16}{3}\pi}$$

(17) Evaluate $\iiint_E \sqrt{x^2+y^2}\, dV$, where E is the region that lies inside the cylinder $x^2+y^2=16$ and between the planes $z=-5$ and $z=4$.

First, sketch the region E



SECOND, SET up the integral
Using cylindrical coordinates:

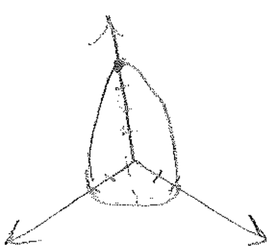
$$E=\left\{(r,\theta,z):\ \begin{array}{l} -5\le z\le 4 \\ 0\le r\le 4, \\ 0\le\theta\le 2\pi \end{array}\right\}$$

$f(x,y,z)=\sqrt{x^2+y^2}=\sqrt{r^2}=r$

$$\int_{-5}^{4}\int_{0}^{2\pi}\int_{0}^{4} r^2\,dr\,d\theta\,dz = \int_{-5}^{4}\int_{0}^{2\pi}\left[\frac{r^3}{3}\right]_0^4 d\theta\,dz = \int_{-5}^{4}\int_{0}^{2\pi}\frac{64}{3}\,d\theta\,dz = \frac{64}{3}\int_{-5}^{4}\int_{0}^{2\pi} d\theta\,dz$$

$$=\frac{64}{3}\int_{-5}^{4}2\pi\,dz = \frac{128}{3}\pi\int_{-5}^{4}dz = \frac{128\pi}{3}(4-(-5)) = \frac{128}{\cancel{3}}\pi\cdot 9 = 3\times 128\pi = \boxed{384\pi}$$

(19) Evaluate $\iiint_E (x+y+z)\,dV$, where E is the solid in the first octant that lies under the paraboloid $z=4-x^2-y^2$



$$E=\left\{(r,\theta,z):\ \begin{array}{l} 0\le\theta\le\pi/2 \\ 0\le r\le 2 \\ 0\le z\le 4-r^2 \end{array}\right\}$$

$$\int_0^{\pi/2}\int_0^{2}\int_0^{4-r^2} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

$$=\iiint(r\cos\theta+r\sin\theta+z)\,r\,dz\,dr\,d\theta = \iiint r^2\cos\theta + r^2\sin\theta + rz\ dz\,dr\,d\theta$$

$$=\iint\left(r^2\cos\theta\, z + r^2\sin\theta\, z + \frac{rz^2}{2}\right)_0^{4-r^2} dr\,d\theta = \iint r^2\cos\theta(4-r^2)+r^2\sin\theta(4-r^2) + \frac{r(4-r^2)^2}{2}$$

$$=\iint 4r^2\cos\theta - r^4\cos\theta + 4r^2\sin\theta - r^4\sin\theta + \frac{r}{2}(16-8r^2+r^4)$$

$$=\iint 4r^2\cos\theta - r^4\cos\theta + 4r^2\sin\theta - r^4\sin\theta + 8r - 4r^3 + \frac{r^5}{2}\ dr\,d\theta$$

$$\int \left(\frac{4\cos\theta}{3} r^3 - \frac{r^5}{5}\cos\theta + \frac{4\sin\theta}{3} r^3 - \frac{r^5}{5}\sin\theta + 4r^2 - \frac{4}{3}r^3 + \frac{r^5}{10} \right)_0^2$$

$$= \int \frac{32}{3}\cos\theta - \frac{32}{5}\cos\theta + \frac{32}{3}\sin\theta - \frac{32}{5}\sin\theta + 16 - \frac{32}{3} + \frac{32}{10}$$

$$= \left(\frac{32}{3}\sin\theta - \frac{32}{5}\sin\theta - \frac{32}{3}\cos\theta + \frac{32}{5}\cos\theta + 16\theta - \frac{32}{3}\theta + \frac{32}{10}\theta\right)_0^{\pi/2}$$

$$= \left(\frac{32}{3} - \frac{32}{5} + 8\pi - \frac{32}{6}\pi + \frac{32}{20}\pi\right) - \left(-\frac{32}{3} + \frac{32}{5}\right)$$

$$= \frac{32}{3} + \frac{32}{3} - \frac{32}{5} - \frac{32}{5} + 8\pi - \frac{32}{6}\pi + \frac{32}{20}\pi$$

$$= \frac{64}{3} - \frac{64}{5} + \frac{16\pi}{6} + \frac{32}{20}\pi = \frac{5\times 64 - 3\times 64}{15} + \frac{160\pi + 96\pi}{60}$$

$2\cdot 3$ $\quad 2\cdot 2\cdot 5$

$$= \frac{128}{15} + \frac{256}{60}\pi = \frac{512 + 256\pi}{60}$$

$$\frac{128}{30} = \frac{64}{15} =$$

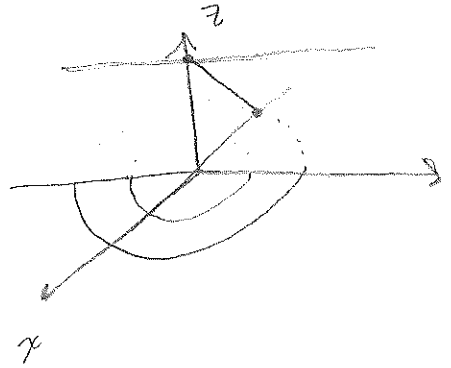
Evaluate the triple integral

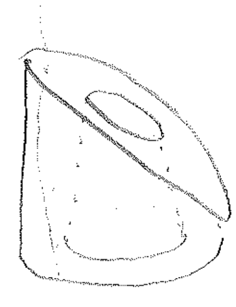
$$\iiint_E y\,dV$$

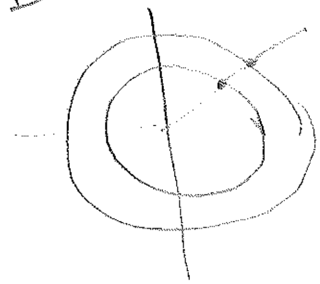
where E is the solid that lies between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$, above the xy-plane and below the plane $z=x+2$

D

E





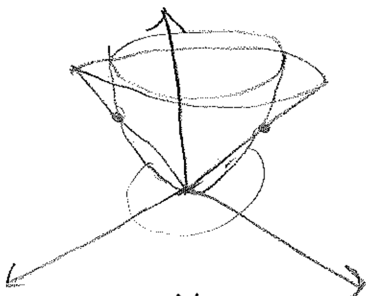


$$E:\left\{(r,\theta,z):\ \begin{array}{l} 0\le\theta\le2\pi, \\ 1\le r\le 2, \\ 0\le z\le r\cos(\theta)+2 \end{array}\right\}$$

$$\int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2}(r\sin\theta)r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2}r^2\sin\theta\,dz\,dr\,d\theta$$

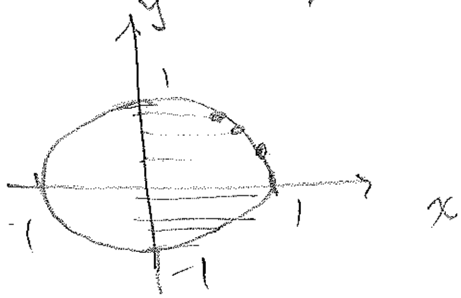
$$=\int_0^{2\pi}\int_1^2 r^2\sin\theta\,(r\cos\theta+2)\,dr\,d\theta=\int_0^{2\pi}\int_1^2$$

0) Convert $\displaystyle\iiint_E = \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\, dz\,dx\,dy$ to cylindrical coordinates



$z_{low} = x^2+y^2$

$z_{high} = \sqrt{x^2+y^2} \iff z^2 = x^2+y^2$



$x_{low} = 0$

$x_{high} = \sqrt{1-y^2} \iff x^2 = 1-y^2$

$\iff x^2+y^2 = 1$

change $\rightsquigarrow xyz \rightsquigarrow r^2 \sin\theta \cos\theta\, z = f(r\cos\theta, r\sin\theta, z)$

$z_{low} = x^2+y^2 \rightsquigarrow z_{low} = r^2$ | $x_{low} = 0 \rightsquigarrow r = 0$

$z_{high} = \sqrt{x^2+y^2} \rightsquigarrow z_{high} = r$ | $x_{high} = \sqrt{1-y^2} \rightsquigarrow r = 1$

$y_{low} = -1 \rightsquigarrow \theta = 0$

$y_{high} = 1 \rightsquigarrow \theta = \pi/2$

the integral in cylindrical coordinates is:

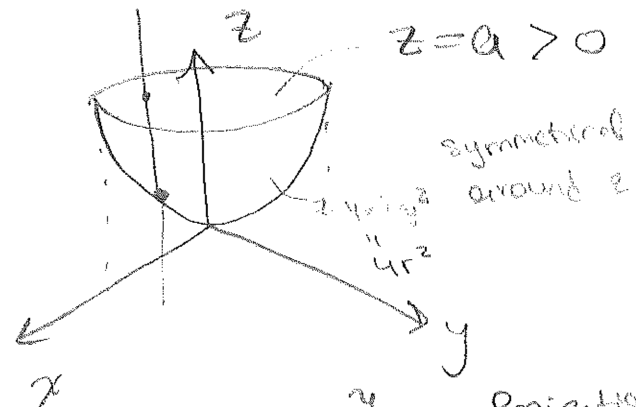
$$\iiint_E f(x,y,z)\,dV = \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{r^2}^{r} (r^2 \sin\theta \cos\theta\, z)\, r\, dz\,dr\,d\theta$$

$$\boxed{= \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{r^2}^{r} r^3 \sin\theta \cos\theta\, dz\,dr\,d\theta}$$

$$\text{mass} = \iiint_E p(x,y,z)\,dv \quad ; \quad p(x,y,z) = \text{density on } (x,y,z)$$

Suppose $p(x,y,z) = K$; where K constant.

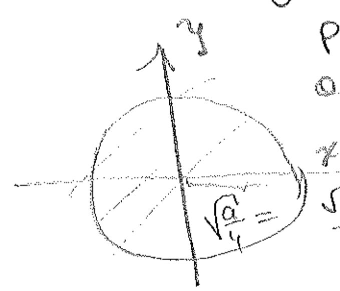
Find the <u>mass</u> and center of mass of the solid E bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a$ $(a > 0)$ if E has constant density K.



mass = density x volume

$$m = \iiint_E p(x,y,z)\,dv$$

$$E = \left\{ (r,\theta,z) \;\middle|\; \begin{array}{l} 0 \le \theta \le 2\pi, \\ 0 \le r \le \frac{\sqrt{a}}{2}, \\ 4r^2 \le z \le a \end{array} \right\}$$



Projection of bounding curve on xy plane

$$m = K\int_0^{\frac{\sqrt{a}}{2}}\int_0^{2\pi}\int_{4r^2}^{a} r\,dz\,d\theta\,dr = K\int_0^{\frac{\sqrt{a}}{2}}\int_0^{2\pi} \underbrace{r(a-4r^2)}_{ar-4r^3}\,d\theta\,dr$$

$$z = 4x^2 + 4y^2 = a$$

$$\Rightarrow x^2 + y^2 = \frac{a}{4}$$

$$= K2\pi\int_0^{\sqrt{a}/2} ar - 4r^3\,dr = K2\pi\left(\frac{ar^2}{2} - r^4\right)\Big|_0^{\sqrt{a}/2}$$

$$= K2\pi\left(\frac{\frac{a^2}{4}}{2} - \frac{a^2}{16}\right) = 2\pi K\left(\frac{a^2}{8} - \frac{a^2}{16}\right) = 2\pi K\left(\frac{a^2}{16}\right)$$

$$= \boxed{\frac{a^2\pi K}{8}}$$

Center of mass $(\bar{x}, \bar{y}, \bar{z})$

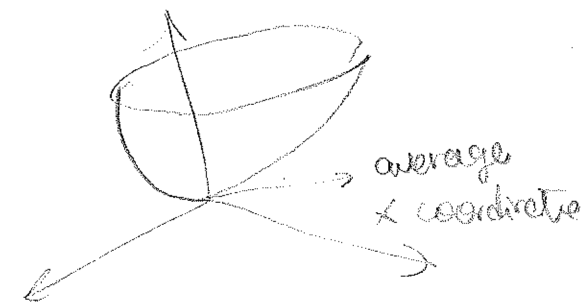
$$\bar{x} = \frac{1}{m} \iiint_{E} x \cdot \overset{K}{\overset{\parallel}{\rho(x,y,z)}} \, dV$$ weigthing the x coordinate

$$K \cdot \frac{1}{m} \int_0^{\sqrt{a}/2} \int_0^{2\pi} \int_{4r^2}^{a} \underbrace{r^2 \cos\theta}_{r \cdot x} \, dz \, d\theta \, dr = \frac{0}{m} = 0$$

same with $\bar{y}, \bar{z}$

$$\int_0^{2\pi} \cos\theta = 0$$ same for $\bar{y}$

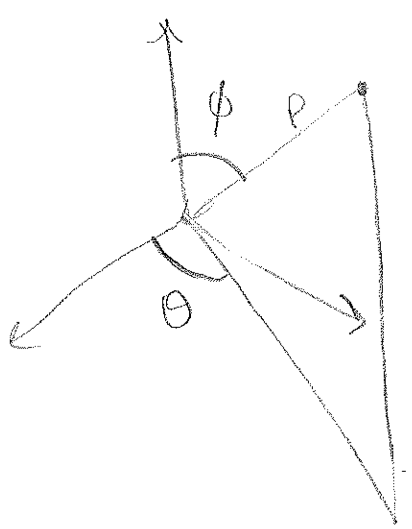
So $\bar{x} = \bar{y} = 0$

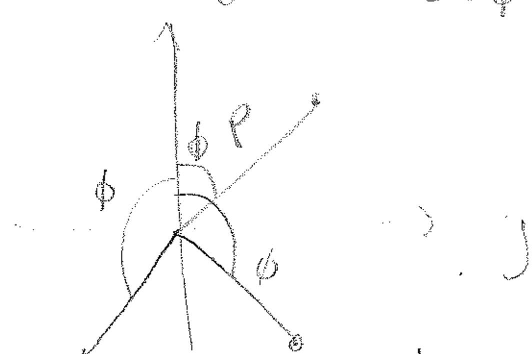


$$\bar{z} = \frac{1}{m} \iiint_{E} z \cdot \rho(x,y,z) \, dV$$ (this is not zero!)

Spherical Coordinates

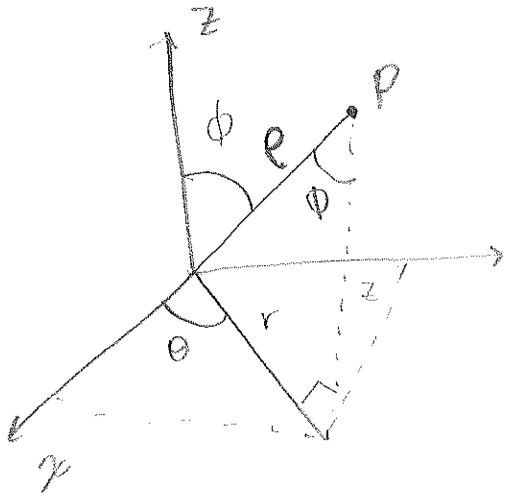
$\rho \geq 0$





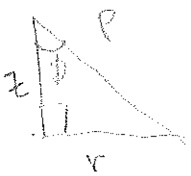
$0 \leq \phi \leq \pi$

Angle from the positive z-axis



$$\frac{r}{\rho} = \sin\phi \Rightarrow r = \rho\sin\phi$$

$$\frac{z}{\rho} = \cos\phi \Rightarrow z = \rho\cos\phi$$



$\sin = \frac{OP}{H} = \frac{r}{\rho}$

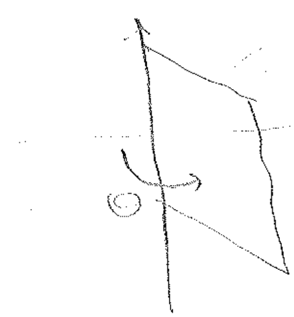
$\cos = \frac{AD}{H} = \frac{z}{\rho}$

Cylindical	Spherecial
$x = r\cos\theta$	$= \rho\sin(\phi)\cdot\cos(\theta)$
$y = r\sin\theta$	$= \rho\sin(\phi)\sin(\theta)$
$z = z$	$= \rho\cos(\phi)$
$r\,dz\,dr\,d\theta$	$(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$

$$\rho^2 = x^2 + y^2 + z^2$$

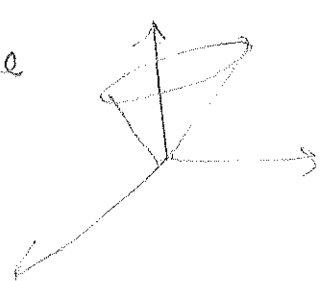
ρ = constant

θ = constant



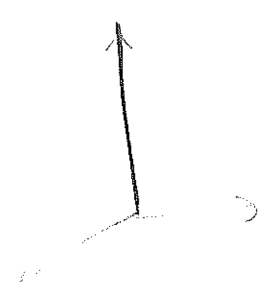
$\phi = c$, a half-cone

$0 < c < \pi/2$

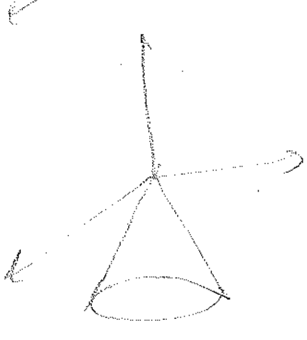


$\phi = \pi/2 \Rightarrow$ xy-plane

$\phi = 0$



$\pi/2 < c < \pi$



$$dv = (\rho^2\sin(\phi))\,d\rho\,d\phi\,d\theta$$

In cylindrical coordinates $E = \left\{ (r,\theta,z) \,\middle|\, \begin{array}{l} \theta_1 \le \theta \le \theta_2 ; \\ a \le r \le b ; \\ g_1(r,\theta) \le z \le g_2(r,\theta) \end{array} \right\}$

$$\iiint_E f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_a^b\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

In Spherical Coordinates $E = \left\{ (\rho,\theta,\phi) \,\middle|\, \begin{array}{l} a \le \rho \le b ; \\ \alpha \le \theta \le \beta ; \\ c \le \phi \le d \end{array} \right\}$

$$\iiint_E f(x,y,z)\,dV = \int_c^d\int_\alpha^\beta\int_a^b f(\rho\sin(\phi)\cos(\phi), \rho\sin(\phi)\sin(\phi), \rho\sin(\phi))\,\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

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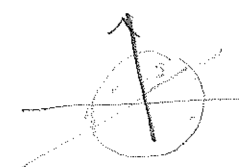
(21) Use spherical coordinates

Evaluate $\iiint_B (x^2+y^2+z^2)^2\,dV$, where B is the ball with center the origin and radius 5.



$x^2+y^2+z^2 = \rho^2$

$$B = \left\{ (\rho,\theta,\phi) \,\middle|\, \begin{array}{l} 0 \le \rho \le 5 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{array} \right\}$$

$$\iiint_B (x^2+y^2+z^2)^2\,dV = \int_0^\pi\int_0^{2\pi}\int_0^5 (\rho^4)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^\pi\int_0^{2\pi} \sin\phi\left[\frac{\rho^7}{7}\right]_0^5 d\theta\, d\phi = \int_0^\pi \sin\phi\left(\frac{5^7}{7}\right)\int_0^{2\pi} d\theta\, d\phi = \frac{5^7}{7}\cdot 2\pi\int_0^\pi \sin\phi\, d\phi$$

$$\frac{(2\pi)5^7}{7}\left(-\cos\phi\right)_0^\pi = \frac{(2\pi)5^7}{7}\left(-\cos\pi + \cos 0\right) = \frac{(2\pi)5^7}{7}(2) = \frac{4\cdot 5^7\pi}{7}$$

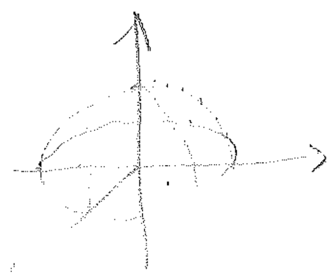
$$= \boxed{\frac{312500\pi}{7}}$$

Find the mass and center of mass of a solid hemisphere of radius a if the density at any point is proportional to its distance from the base.

First, remember mass = density × volume.

In this case density = $d(x,y,z) = K \cdot z$

$$E = \left\{ (\rho,\theta,\phi) \mid \begin{array}{l} 0 \le \rho \le a, \\ 0 \le \theta \le 2\pi, \\ 0 \le \phi \le \pi/2 \end{array} \right\}$$



$$\text{mass} = \iiint_E d(x,y,z)\,dv$$, In spherical coordinates

$$\rightarrow \text{mass} = \int_0^{\pi/2}\int_0^{2\pi}\int_0^{a} (K\rho\cos\phi)(\rho^2\sin\phi)\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{2\pi}\int_0^{a} K\rho^3\cos\phi\sin\phi\,d\rho\,d\theta\,d\phi = 2\pi\cdot K\left(\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\right)\left(\int_0^{a}\rho^3\,d\rho\right)$$

$$= 2\pi K\left(\frac{\sin^2\phi}{2}\right)\Bigg|_0^{\pi/2}\left(\frac{\rho^4}{4}\right)\Bigg|_0^{a} = \boxed{\frac{K\pi a^4}{4}}$$

For center of mass. $\bar{x} = \dfrac{\iiint_E x\,\rho(x,y,z)\,dv}{\text{mass}}$; $\bar{y} = \dfrac{\iiint_E y\,\rho(x,y,z)\,dv}{\text{mass}}$

$\bar{z} = \dfrac{\iiint_E z\,\rho(x,y,z)\,dv}{\text{mass}}$

Notice that symmetry inmediately shows the x, y coordinates of the center of mass are 0.