M311 Final Exam

Spring 2013

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Circle class time: 10:10 11:15 (Recitation)

1. How far from the origin O = *(0, 0, 0)* is the plane $3x + 2y + z = 6$?

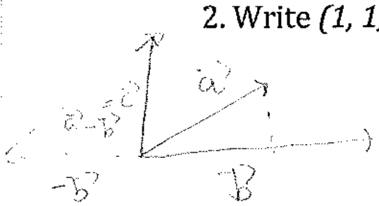
the normal vector to the plane is $(3,2,1)$.

Let P be a point on the plane, e.g. $P(1,1,1)$.

then, $\vec{OP} = P - O = \vec{P}$. Project $\vec{P}$ onto $\vec{n}$.

$$\text{Comp}_{\vec{n}} \vec{P} = \frac{\vec{P}\cdot\vec{n}}{|\vec{n}|} = \frac{\langle 1,1,1\rangle \cdot \langle 3,2,1\rangle}{\sqrt{9+4+1}} = \frac{3+2+1}{\sqrt{14}} = \boxed{\frac{6}{\sqrt{14}}}$$

2. Write *(1, 1, 1)* as the sum of vectors parallel to and orthogonal to *(1, 0, 2)*.



$$(1,1,1) = \text{Parallel} + \text{orthogonal}$$
$$= \text{Proj}_{(1,0,2)}(1,1,1) + \left((1,1,1) - \text{Proj}_{(1,0,2)}(1,1,1)\right),$$

where, $\text{Proj}_{(1,0,2)}(1,1,1) = \frac{(1,0,2)}{|(1,0,2)|} \cdot \text{Comp}_{(1,0,2)}(1,1,1) = \frac{1+2}{1^2+2^2} \cdot (1,0,2) = \frac{3}{5}(1,0,2)$

Hence, $(1,1,1) = \frac{3}{5}(1,0,2) + \left((1,1,1) - \frac{3}{5}(1,0,2)\right) = \underbrace{\frac{3}{5}(1,0,2)}_{\text{Parallel}} + \underbrace{\left(\frac{2}{5},1,-\frac{1}{5}\right)}_{\text{orthogonal}}$

Since: $\left(\frac{2}{5},1,-\frac{1}{5}\right)\cdot(1,0,2) = \frac{2}{5} - \frac{2}{5} = 0$

$\Leftrightarrow \left(\frac{2}{5},1,-\frac{1}{5}\right) \perp (1,0,2)$

3. Write *(1, 1, 1)* as the sum of vectors parallel to and orthogonal to the plane

$3x + 2y + z = 6.$

An orthogonal vector to the plane is $\vec{n} = (3,2,1)$.

A vector parallel to the plane is.

4. Find and classify the critical points of $f(x, y) = x^4 + y^4 - 4xy$.

$$\frac{\partial f}{\partial x} = 4x^3 - 4y = 0 \Rightarrow x^3 = y \quad ; \qquad \frac{\partial^2 f}{\partial x^2} = 12x^2$$

$$\frac{\partial f}{\partial y} = 4y^3 - 4x = 0 \Rightarrow y^3 = x \qquad \frac{\partial^2 f}{\partial y^2} = 12y^2$$

Critical points : $x = 0 \Rightarrow y = 0$ $\quad (0,0)$

$x = 1 \Rightarrow y = 1$ $\quad (1,1)$

$x = -1 \Rightarrow y = -1$ $\quad (-1,-1)$

$$D = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{vmatrix} = 144x^2y^2$$

$D(0,0) = 0 \Rightarrow$ Saddle point

$D(1,1) = 144 > 0 \Rightarrow \frac{\partial^2 f}{\partial x^2}(1,1) = 12 > 0 \Rightarrow (1,1)$ is a local min

$D(-1,-1) = 144 > 0 \Rightarrow \frac{\partial^2 f}{\partial y^2}(-1,-1) = 12 > 0 \Rightarrow (-1,-1)$ is a local min

5. Use Lagrange multipliers to find the maximum value of $f(x, y) = xy$ if $x^2 + 4y^2 = 4$, $x \geq 0$.

$$\nabla f(x,y) = \langle y, x \rangle = \lambda \nabla g = \lambda \langle 2x, 8y \rangle$$

$$\Rightarrow \langle y, x \rangle = \lambda \langle 2x, 8y \rangle$$

$$\begin{cases} y = \lambda 2x \Rightarrow \frac{y}{2x} = \lambda \\ x = \lambda 8y \end{cases} \Downarrow \quad x = \left(\frac{y}{2x}\right) 8y = 2x^2 = 8y^2$$

$$x^2 = 4y^2$$

$\Rightarrow$ Since $x^2 + 4y^2 = 4 \Rightarrow 4y^2 + 4y^2 = 4 \Rightarrow 8y^2 = 4 \Rightarrow y^2 = \frac{1}{2}$

$\boxed{y = \pm \frac{1}{\sqrt{2}}} \Rightarrow x^2 + 4\frac{1}{2} = 4 \Rightarrow x^2 = 2 \Rightarrow \boxed{x = \pm\sqrt{2}}$

the points are $\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right); \left(-\sqrt{2}, \frac{1}{\sqrt{2}}\right), \left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right); \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right)$

the maximum value is $f\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right) = \sqrt{2} \cdot \frac{1}{\sqrt{2}} = \boxed{1}$ at $\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right)$

If $(x,y)=(1,0)$ then
$z=1 \Rightarrow (x,y,z)=(1,0,1)$

6. Give an equation for the tangent plane to $z = x^4 + y^4 - 4xy$ at $(x, y) = (1, 0)$.

Define $g(x,y,z) = x^4 + y^4 - 4xy - z$. the gradient of this function defines a vector orthogonal to its level surfaces, i.e., to $z = x^4 + y^4 - 4xy$. Hence, the normal vector for the plane we want is $\vec{n} = \nabla g(1,0,1) = \langle 4x^3 - 4y, 4y^3 - 4x, -1 \rangle$
$\Rightarrow \vec{n} = \langle 4, -4, -1 \rangle$ the equation for the plane with normal vector $\vec{n}$ through the point (x_0, y_0, z_0) is: $\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = 0$

$\Rightarrow \langle 4, -4, -1 \rangle \cdot \langle x-1, y, z-1 \rangle = 0 \Rightarrow 4x - 4 - 4y - z + 1 = 0$

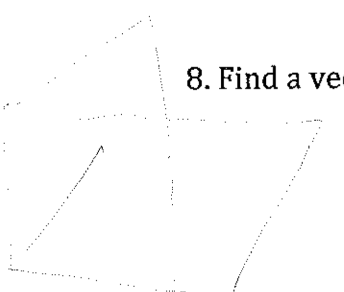
$\Rightarrow$ Tangent plane: $\boxed{4x - 4y - z = 3}$

7. Give an equation for the tangent plane to the level surface of $F(x, y, z) = x^2/4 + y^2 - z^2/9$ at $(x, y, z) = (2, -1, 9)$. In what direction is F increasing most rapidly, and at what rate?

$\nabla F = \langle \frac{x}{2}, 2y, -\frac{2}{9}z \rangle$ Hence, $\vec{n} = \nabla F(2,-1,9) = \langle 1, -2, -2 \rangle$

$\vec{n} \cdot (x-2, y+1, z-9) = 0$
$\langle 1, -2, -2 \rangle \cdot \langle x-2, y+1, z-9 \rangle = 0 = x - 2 - 2y - 2 - 2z + 18 = 0$
$\Rightarrow \boxed{x - 2y - 2z = -14}$ the direction of fastest increase at $(2,-1,9)$ is $\nabla F(2,-1,9) = \langle 1, -2, -2 \rangle$ at the rate $|\nabla f(2,-1,9)| = \sqrt{1+4+4}$ $\boxed{= 3}$

8. Find a vector parallel to the intersection of the two planes from #6 and #7.



$\begin{cases} 4x - 4y - z = 3 \\ x - 2y - 2z = -14 \end{cases} \Rightarrow \begin{cases} 4x - 4y - z = 3 \\ -4x + 8y + 8z = 56 \end{cases}$

$4y + 7z = 59$

9. Find the distance between the point *(1, 0, 2)* and the plane $3x + 2y + z = 6$.

10. Find the distance between the point *(1, 0, 2)* and the line

$r(t) = (1, 1, 1) + t\,(1, 1, 2)$.

$$d(r(t), P)^2 = d((1+t, 1+t, 1+2t), (1,0,2))^2$$
$$= t^2 + (1+t)^2 + (2t-1)^2$$
$$= t^2 + t^2 + 2t + 1 + 4t^2 - 4t + 1$$
$$= 6t^2 - 2t + 2$$

We want to minimize $d(t) = 6t^2 - 2t + 2$.

$d'(t) = 12t - 2 = 0 \Rightarrow t = \frac{2}{12} = \boxed{\frac{1}{6}}$

$d''(t) = 12 \Rightarrow \boxed{t = \frac{1}{6}}$ is the minimum.

$r(\frac{1}{6}) = (1,1,1) + \frac{1}{6}(1,1,2) = (\frac{7}{6}, \frac{7}{6}, \frac{4}{3})$

$$d((\tfrac{7}{6}, \tfrac{7}{6}, \tfrac{4}{3}), (1,0,2)) = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{7}{6}\right)^2 + \left(\frac{4}{6}\right)^2}$$
$$= \sqrt{\frac{1+49+16}{36}} = \sqrt{\frac{66}{36}} = \boxed{\frac{\sqrt{66}}{6}}$$

11. Find the length of the curve ***r(t)*** *= (1, t², t³)*, for *0 ≤ t ≤ 1*.

$$L = \int_0^1 \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt = \int_0^1 \sqrt{0^2+(2t)^2+(3t^2)^2}\,dt$$

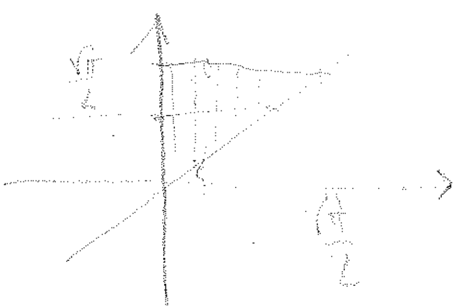
$$= \int_0^1 \sqrt{4t^2+9t^4} = \int_0^1 \sqrt{t^2(4+9t^2)} = \int_0^1 t\cdot\sqrt{4+9t^2}\,dt$$

$u = 4+9t^2 \Rightarrow du = 18t\,dt \rightsquigarrow \int_\square^\square \frac{du}{18}\sqrt{u} = \frac{1}{18}\int_\square^\square \sqrt{u}\,du = \frac{1}{18}\cdot\frac{2}{3}u^{3/2}$

$\Rightarrow \frac{du}{18} = t\,dt$

$$= \frac{1}{27}(4+9t^2)^{3/2}\Big|_0^1 \rightsquigarrow \frac{1}{27}\left[(13)^{3/2} - 4^{3/2}\right]$$

12. Evaluate $\int_0^{\sqrt{\pi/4}}\int_x^{\sqrt{\pi/4}} \sin(y^2)\;dy\,dx$.



$$\int_0^{\sqrt{\frac{\pi}{4}}}\int_x^{\sqrt{\frac{\pi}{4}}} \sin(y^2)\,dy\,dx = \int_0^{\sqrt{\frac{\pi}{4}}}\int_0^{y} \sin(y^2)\,dx\,dy$$

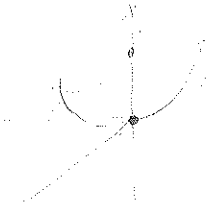
$$= \int_0^{\sqrt{\frac{\pi}{4}}} \sin(y^2)\,[x]_0^y\,dy = \int_0^{\sqrt{\frac{\pi}{4}}} \sin(y^2)\,y\,dy$$

$y^2 = u \Rightarrow du = 2y\,dy$

$\Rightarrow \frac{du}{2} = y\,dy \rightsquigarrow \int_\square^\square \sin(u)\frac{du}{2} = \frac{1}{2}\int_\square^\square \sin(u)\,du$

$$= \frac{1}{2}\left[-\cos(u)\right]_\square^\square \rightsquigarrow \frac{1}{2}\left[-\cos(y^2)\right]_0^{\sqrt{\frac{\pi}{4}}} = \frac{1}{2}\left[-\cos\left(\frac{\pi}{4}\right)+\cos(0)\right]$$

$$= \frac{1}{2}\left[\cos(0) - \cos\left(\frac{\pi}{4}\right)\right] = \frac{1}{2}\left[1-\frac{\sqrt{2}}{2}\right] = \frac{1}{2}\left[\frac{2-\sqrt{2}}{2}\right] = \boxed{\frac{2-\sqrt{2}}{4}}$$



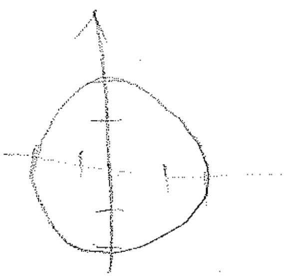
$4x^2+4y^2+z^2=16$

13. Use cylindrical coordinates to find the volume of $4x^2 + 4y^2 + z^2 \le 16$.

cylindrical coordinates: $x = r\cos(\theta)$; $y = r\sin(\theta)$; $z = z$.

$z=0 \Rightarrow 4x^2+4y^2=16 \Rightarrow x^2+y^2=2^2$

the domain is



$$\int_0^{2\pi}\int_0^{2}\int_0^{4} 1\cdot dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} 4\,dr\,d\theta = \int_0^{2\pi} 8\,d\theta = \boxed{16\pi}$$

14. Rewrite in rectangular coordinates: $$\int_0^{\pi}\int_0^{\pi/2}\int_0^{2} \rho^2 \sin(\phi)\,d\rho\,d\phi\,d\theta.$$